# SPLINE METHODS FOR THE NUMERICAL SOLUTION OF SINGULAR PERTURBATION PROBLEMS

*A Thesis Submitted*

in Partial Fulfilment of the Requirements

for the Degree of

Doctor of Philosophy

*by*

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*to the*

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

MAY, 2001

# CERTIFICATE

It is certified that the work contained in the thesis entitled **"Spline Methods for the Numerical Solution of Singular Perturbation Problems"**, by **Kailash C. Patidar (Roll No: 9610876)**, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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# Acknowledgement

It gives me immense pleasure to have the priviledge of extending my deepest sense of gratitude and sincere thanks to my supervisor **Prof. Mohan K. Kadalbajoo**, for his invaluable guidance, constructive criticism, excellant rapport and willingness to discuss the topic at many odd hours throughout the course of this work. His expertise skill and constant support are major pluses for this research. I owe great debt to his wife Mrs. Usha Kadalbajoo for inspiring discussions and moral support. Thanks are also due to his daughters Miss Mridula and Miss Bhavna for keeping lots of patience during my disscussion with Dr. Kadalbajoo.

I wish to express my gratitude to Prof. R.K.S. Rathore, Prof. Manjul Gupta, Prof. Praval Sinha, Prof. V. Raghvendra, Prof. Prabha Sharma, Dr. D. Kundu, Dr. B.V. Rathish Kumar, Dr. D. Bahuguna, Dr. Alok Maloo, Dr. R. Tiwari (CSE) and Dr. Sushma Tiwari and other faculties of the Mathematics Department at IIT Kanpur.

I am also thankful to Prof. R.B. Kellogg, University of Maryland, College Park, Maryland; Prof. P.W. Hemker, Mathematisch Centrum, Amsterdam; Prof. J.E. Flaherty, Rensselaer Polytechnic Institute, New York and Prof. M.K. Jain (Ret. Prof.), University of Mauritius, for many fruitful discussions on different aspects of the problems considered in this thesis.

Thanks are due to Prof. C. L. Parihar and Prof. M.K. Dube of Holkar Science College, Indore, M.P.; Dr. S.C.S. Rao, Asst. Prof., IIT Delhi and Dr. A. Singh, Asst. Prof., IIT Chennai, for their invaluable guidence as and when required.

I would like to give special thanks to Dr. Manoj T. Nair, Scientist, CTFD Division, National Aerospace Laboratories, Bangalore, for all kind of helps (since the begining and till the finishing touch of this work). Specially he provided me maximum of the important research material and helped me a lot in essential programming.

I owe my deep sense of gratitude to my grandparents, parents, all other relatives, AMG Sarangi, families and friends at IIT Kanpur.

Finally, I am pleased to thank to my dearest brother Dilip, my brother-in-law Dr. Sunil Goria and my wife Pramila for providing me the constant help, moral support and encouraging environment to take the challenges.

**Kailash C. Patidar**

*Dedicated*

*to*

*Goddess Durga*

*and*

*Lord Uchchhishthha Ganesh*

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# Publications:

A part of this thesis has been accepted/submitted in the form of following research papers:

1. Numerical Solution of Singularly Perturbed Two Point Boundary Value Problems by Spline in Compression, **International Journal of Computer Mathematics**, Vol. 77, no. 2, 263-284, 2001.
2. Numerical Solution of Singularly Perturbed Non-Linear Two Point Boundary Value Problems by Spline in Compression, to appear in **International Journal of Computer Mathematics.**
3. Spline Approximation Method for Solving Self-adjoint Singular Perturbation Problems on Non-Uniform Grids, to appear in **Journal of Computational Analysis and Applications.**
4. Variable Mesh Spline Approximation Method for Solving Singularly Perturbed Turning Point Problems Having Boundary Layer(s), to appear in **Computers and Mathematics with Applications.**
5. Spline Techniques for the Numerical Solution of Singular Perturbation Problems, to appear in **Journal of Optimization Theory and Applications.**
6. Numerical Solution of Singularly Perturbed Two Point Boundary Value Problems by Spline in Tension, to appear in **Applied Mathematics and Computation.**
7. A Survey of Numerical Techniques for Solving Singulalry Perturbed Ordinary Differential Equations, to appear in **Applied Mathematics and Computation.**
8. Tension Spline for the Solution of Self-adjoint Singular Perturbation Problems, submitted for publication.
9. Exponentially Fitted Spline in Compression for the Numerical Solution of Singular Perturbation Problems, submitted for publication.
10. Exponentially Fitted Spline Approximation Method for Solving Self-adjoint Singular Perturbation Problems, submitted for publication.
11. Variable Mesh Spline in Compression for the Numerical Solution of Singular Perturbation Problems, submitted for publication.
12. Tension Spline for The Numerical Solution of Singularly Perturbed Non-Linear Boundary Value Problems, submitted for publication.
13. Variable Mesh Spline Approximation Method for Solving Singularly Perturbed Turning Point Problems Having Interior Layer, submitted for publication.
14. Spline Techniques for Solving Singularly Perturbed Non-Linear Problems on Non-Uniform Grids, submitted for publication.

# SYNOPSIS

The theory of singular perturbations is not a settled direction in mathematics and the path of its development is a dramatic one. In the intensive development of science and technology, many practical problems, such as the mathematical boundary layer theory or approximation of solutions of various problems described by differential equations involving large or small parameters, become more complex, and, therefore, in their analysis it is natural to use asymptotic methods. However, the asymptotic analysis for differential operators has a developed theory mainly for the case of regular perturbations, when the perturbations carry a subordinate character with respect to the unperturbed operator. In some problems the perturbations are operative over a very narrow regions across which the dependent variable undergo very rapid changes. These narrow regions frequently adjoin the boundaries of the domain of interest, owing to the fact that the small parameter multiplies the highest derivative. Consequently, they are usually referred to as boundary layers in fluid mechanics, edge layers in solid mechanics, skin layers in electrical applications, shock layers in fluid and solid mechanics, transition points in quantum mechanics and Stokes lines and surfaces in mathematics.

Numerical analysis and asymptotic analysis are two principal approaches for solving singular perturbation problems. Since the goals and the problem classes are rather different, there has not been much interaction between these approaches. Numerical analysis tries to provide quantitative information about a particular problem, whereas asymptotic analysis tries to gain insight into the qualitative behaviour of a family of problems and only semiquantitative information about any particular member of the family. Numerical methods are intended for a broad classes of problems and are intended to minimize demands upon the problem solver. Asymptotic methods treat comparatively restricted classes of problems and require the problem solver to have some understanding of the behaviour of the solution expected.

The present thesis deals with the numerical methods for solving singular perturbation problems (SPPs). This thesis is divided into seven chapters. We consider mainly the one dimensional singularly perturbed second order two point boundary value problems. Methods are devised for linear problems, nonlinear problems, turning point problems having boundary layers and the turning point problems having interior layers. The methods developed have been analysed for convergence and it has been found that all

the methods are second order accurate. Several numerical experiments are given at the end of each chapter.

The first chapter reviews in detail the various numerical methods to solve the one-dimensional SPPs. This review contains a surprisingly large amount of material and indeed can serve as an introduction to some of the ideas and methods of singular perturbation theory. Starting from Prandtl's work a large amount of work has been done in the area of singular perturbations. This chapter limits its coverage to some standard singular perturbation models considered by various workers, the numerical methods developed by numerous researchers to solve SPPs and to the summary of this thesis.

Consider the following class of SPPs:

$$\left.\begin{array}{l} Ly \equiv \varepsilon y'' + a(x)y' + b(x)y = f(x) \\ y(0) = \alpha_0 \ , \ y(1) = \alpha_1 \ ; \ \alpha_0, \alpha_1 \in R \end{array}\right\} \tag{1}$$

where, $a(x)$, $b(x)$, $f(x)$ are sufficiently smooth with $a(x) > c > 0$, $b(x) > 0$, $c$ is some constant and $\varepsilon$ is a small positive parameter.

It is well known that the solution of the problem (1.1) when $a(x) \equiv 0$ has oscillatory behaviour. To overcome these oscillations, in chapter 2, we develop a method using spline in compression. In this chapter we consider three types of problems. First we analyse the problems in which the second derivative term and the function term in (1.1) are present while the term containing the first derivative is absent. The problems having the second and first derivative terms but lacking the function term are considered in the second case. Finally the third case deals with the most general problem. By making use of the continuity of the first order derivative of the spline function, the resulting spline difference scheme gives a tridiagonal system which can be solved efficiently by the well known algorithms.

For the problems of the type

$$\left.\begin{array}{l} Ly \equiv -\varepsilon y'' + a(x)y' + b(x)y = f(x) \quad \text{on} \quad (0,1) \\ y(0) = \alpha_0, \ y(1) = \alpha_1, \ \alpha_0, \alpha_1 \in R \end{array}\right\} \tag{2}$$

where, $x \in (0,1)$, $\varepsilon$ is a small positive parameter and $a(x)$, $b(x)$ and $f(x)$ are bounded continuous functions, we give another method which is based on spline in tension. It is given in chapter 3.

There are two possibilities to obtain small truncation error inside the boundary layer(s). The first is to choose a fine mesh there whereas the second one is to choose a difference formula reflecting the behaviour of the solution(s) inside the boundary layer(s). Since the spline difference scheme has the same order of precision and the same matrix

structure on the uniform and on the non-uniform grid for a fixed $\varepsilon$, we use this property for singularly perturbed problems. This enables us in modifying the distribution of mesh points vis-a-vis to the properties of the exact solution. Keeping these two ideas in mind, in chapter 4, we give two other methods: one is Variable Mesh Spline in Compression referred to as VMSC and the other is Exponentially Fitted Spline in Compression referred to as EFSC to solve the problems of the type (1 ) when $b(x) \equiv 0$. It has also been explained in the same chapter that why the methods VMSC and EFSC have not been used for the case when both $a(x)$ and $b(x)$ in (1 ) are non-zero.

In chapter 5, we consider the following class self-adjoint singularly perturbed two point boundary value problems

$$\left.\begin{array}{l} Ly \equiv -\varepsilon\,(a(x)y')' + b(x)y = f(x) \quad \text{on} \quad (0,1) \\ y(0) = \eta_0, \quad y(1) = \eta_1 \end{array}\right\} \tag{3}$$

where, $\eta_0$, $\eta_1$ are given constants and $\varepsilon$ is a small positive parameter. Further, the coefficients $f(x)$, $a(x)$ and $b(x)$ are smooth functions and satisfy

$$a(x) \geq a > 0,\ a'(x) \geq 0,\ b(x) \geq b > 0$$

The original problem (i.e. problem (3..)) is reduced to the normal form. We then apply spline in tension to the normal form. To obtain the improved results over this method we give two other methods: one is Variable Mesh Cubic Spline method referred to as VMCS and the other is Exponentially Fitted Cubic Spline method referred to as EFCS. Both these methods are presented in this chapter along with the reason for not using the Spline in Tension with variable mesh/exponential fitting for such problems.

The methods developed in chapters 2 and 3 for linear problems are extended for nonlinear problems in chapter 6. Using the well known quasilinearization method of Bellman and Kalaba (1965), the original nonlinear differential equation has been linearized as a sequence of linear differential equations. Each of these linear equations is then solved by the schemes derived for linear case using spline in compression and spline in tension. (We also use variable mesh cubic spline for some problems). In limit, the solution of these linearized problems converges to the solution of the original nonlinear problem.

Finally, in chapter 7, we consider the following class of singularly perturbed turning point problems

$$\left.\begin{array}{l} Ly \equiv \varepsilon y'' + a(x)y' - b(x)y = f(x) \quad \text{on} \quad [p_1, p_2] \\ y(p_1) = \eta_1, \quad y(p_2) = \eta_2 \end{array}\right\} \tag{4}$$

where, $a(x)$ is assumed to be in $C^2[p_1, p_2]$, $b(x)$ and $f(x)$ are required to be in $C^1[p_1, p_2]$, $\eta_1$, $\eta_2$ are given constants, $p_1 \leq 0$, $p_2 > 0$ (usually $p_1 = -1$ and $p_2 = 1$), $0 < \varepsilon << 1$. Moreover

$$a(0) = 0 \quad , \quad a'(0) < 0 \tag{5}$$

In order that the solution of (1.18) satisfies a maximum principle, we require that

$$b(x) \geq 0 \quad , \quad b(0) > 0 \tag{6}$$

Also $b(x)$ is required to be bounded below by some positive constant $b$, i. e.,

$$b(x) \geq b > 0 \tag{7}$$

to exclude the so-called resonance cases. We also impose the following restriction which ensures that there are no other turning points in the interval $[p_1, p_2]$:-

$$|a'(x)| \geq \left|\frac{a'(0)}{2}\right| \quad , \quad x \in [p_1, p_2] \tag{8}$$

(If there is no first derivative term but if $b(x)$ changes sign then also turning point occurs, conventionally termed as classical turning point).
Under these conditions ( 5-8 ), the turning point problem ( 4 ) has a unique solution having two boundary layers at $x = p_1$ and $x = p_2$.

Using the usual first order Taylor series approximations for the first order derivative of the approximate solution which we seek using Ahlberg's cubic spline, the continuity condition give us a tridiagonal system. The variable mesh strategies for three different cases, viz., when there are two boundary layers (one at both ends), when there is one boundary layer at the left end and when there is one boundary layer at the right end are given in this chapter.

In the above problem instead of ( 5 ) if we have the following condition

$$a(0) = 0 \quad , \quad a'(0) > 0 \tag{9}$$

then the solution of the problem ( 4 ) will possess interior layer. We give the treatment of such problems in the same chapter.

The computations reported in this thesis were done on Silicon Graphics Origin 200 (dual processor) Operating System (in Fortran 77 in double precesion with 16 significant figures) at IIT Kanpur.

# Chapter 1

## GENERAL INTRODUCTION

### 1.1 Introduction

The theory of singular perturbations is not a settled direction in mathematics and the path of its development is a dramatic one. In the intensive development of science and technology, many practical problems, such as the mathematical boundary layer theory or approximation of solutions of various problems described by differential equations involving large or small parameters, become more complex, and, therefore, in their analysis it is natural to use asymptotic methods. However, the asymptotic analysis for differential operators has a developed theory mainly for the case of regular perturbations, when the perturbations carry a subordinate character with respect to the unperturbed operator. In some problems the perturbations are operative over a very narrow regions across which the dependent variable undergo very rapid changes. These narrow regions frequently adjoin the boundaries of the domain of interest, owing to the fact that the small parameter multiplies the highest derivative. Consequently, they are usually referred to as boundary layers in fluid mechanics, edge layers in solid mechanics, skin layers in electrical applications, shock layers in fluid and solid mechanics, transition points in quantum mechanics and Strokes lines and surfaces in mathematics.

Towards the end of $19^{th}$ century, the science of Fluid Mechanics was diverging in two mutually exclusive directions - theoretical hydrodynamics and hydrolics. The former evolved from Euler equations for inviscid flows and achieved a high degree of completeness. Unfortunately, the results obtained by using this so-called classical science stood in glaring contradiction to the experimental results. The famous d'Alembert's Paradox is an illustrative example. This aroused the scientists to develop their own empirical

science of hydrolics which was based primarily on a large number of experimental data. In his paper on "Fluid Motion with Very Small Friction", read before the Mathematical Congress in Heidelberg in 1904, Prandtl proved that the flow about a body can be treated by dividing it into two regions : a very thin layer in proximity to the body (which he called boundary layer) where frictional effects are prominent, and the remaining outside region. On the basis of this hypothesis, Prandtl emphasized the importance of viscous flows without delving into the mathematical complexities involved. This boundary-layer theory became the foundation stone for modern fluid dynamics.

Thus the birth of singular perturbations occured at the Third International Congress of Mathematicians in Heidelberg in 1904. Prandtl's seven page report was contained in the proceedings of the conference [195]. However, the term "singular perturbations" was first used in the work of Friedrichs and Wasow [81], a paper which followed a productive New York University seminar on nonlinear vibrations. The solution of singular perturbation problems typically contains layers. Though Prandtl introduced the terminology *boundary layer* in this conference but it got much greater generality in the substantial work of Wasow [267].

Numerical analysis and asymptotic analysis are two principal approaches for solving singular perturbation problems. Since the goals and the problem classes are rather different, there has not been much interaction between these approaches. Numerical analysis tries to provide quantitative information about a particular problem, whereas asymptotic analysis tries to gain insight into the qualitative behaviour of a family of problems and only semiquantitative information about any particular member of the family. Numerical methods are intended for a broad classes of problems and are intended to minimize demands upon the problem solver. Asymptotic methods treat comparatively restricted classes of problems and require the problem solver to have some understanding of the behaviour of the solution expected. Since the mid-1960's, singular perturbations has flourished. The subject is now commonly part of a graduate students training in applied mathematics and in many fields of engineering. Numerous good textbooks have appeared in this area which either dealt with asymptotic approach or with numerical ones. Some of the books dealt with both of these. The list is quite long but we mention few of them. These include Erdelyi [73], Van Dyke [246], Bellman [23], Kaplaun [130], Carrier and

Pearson [38], Dingle [63], Eckhaus [67], Nayfeh [175], O'Malley [182], Willoughby [269], Van Dyke [247], Brauner et al. [35], Hemker [93], Bender and Orszag [24], Childs et al. [50], Eckhaus [68], Hemker and Miller [94], Hughes [104], Na [173], Verhulst [251], Doolan et al. [66], Meyer and Parter [164], Axelsson et al. [18], Kevorkian and Cole [134], Miranker [170], Nayfeh [176], Eckhaus and de Jager [69], Verhulst [252], Ardema [11], O'Malley [185], Miller [166], Quarteroni and Valli [198], Holmes [99], Kevorkian and Cole [135], Miller et al. [168], Morton [171] and Roos et al. [206].

The present survey contained in this chapter includes the research articles on one dimensional singular perturbation problems which can be devided under the categories given in the following table:

$$-\varepsilon y'' + p(x)y' + q(x)y = g(x), \quad a \leq x \leq b$$
$$y(a) = \alpha, \quad y(b) = \beta$$

| Conditions on $p(x)$ | Type of solution |
|---|---|
| $p(x) \neq 0$ on $a \leq x \leq b$: | |
| $p(x) < 0$ | Boundary layer at $x = a$ |
| $p(x) > 0$ | Boundary layer at $x = b$ |
| $p(x) = 0$: | |
| $q(x) > 0$ | Boundary layers at $x = a$ and $x = b$ |
| $q(x) < 0$ | Rapidly Oscillating Solution |
| $q(x)$ changes sign | Classical turning point |
| $p'(x) \neq q(x)$, $p(0) = 0$: | |
| $p'(0) < 0$ | No boundary layers,<br>interior layer at $x = 0$ |
| $p'(0) > 0$ | Boundary layers at $x = a$ and $x = b$<br>no interior layer at $x = 0$ |

This table gives a summary of variety of linear problems. Apart from these classes of problems (included in the table) the other classes of problems, e.g., nonlinear problems, semilinear problems, quasilinear problems (all one dimensional) are also being surveyed in the present chapter.

The numerical techniques used by various workers in the papers included in this survey can broadly be classified as Finite difference methods, Finite element methods and Spline approximation methods but the survey has not been arranged under these categories. The main reason behind this is that there are some papers which use a combination of more than one of the above mentioned methodologies which may create confusion to the

readers who are interested to see the work which falls under a particular direction out of these categories and at the same time they may not get the full information about that direction even if the concerened paper is being included in this chapter. Therefore, we gave the survey chronologically, from 1985 onwards, according to their appearance in the various standard international journals/conference proceedings. So one can see that from year to year, what are the developments in this area? However, the references are arranged in alphabatical order according to first author's name and not in the order as they appeared in the text.

A brief outline of this chapter is as follows: Section 1.2 is devoted to some standard singular perturbation models which arise in various branches of applied sciences and engineering. The survey on numerical techniques used to solve the singular perturbation problems is given in Section 1.3. Section 1.4 deals with the authors contribution to some singularly perturbed two point boundary value problems. In Section 1.5, we discuss some future developments that need attention.

Finally we would like to remark that we have tried to cover as much information as we have. However, some of the important papers from the BAIL conferences (in which the work was focused on boundary and interior layers) have not been included due to their nonavailability. We would like to appologize if there are any other omissions, which are totally unintentional.

## 1.2 Standard Singular Perturbation Models

The development of small parameter methods led to the efficient use of boundary layer theory in various fields of applied mathematics, for instance, fluid mechanics, fluid dynamics, elasticity, quantum mechanics, plasticity, chemical-reactor theory, aerodynamics, plasmadynamics, magnetohydrodynamics, rarefied-gas dynamics, oceanography, meteorology, diffraction theory, reaction-diffusion processes, nonequilibrium and radiating flows and other domains of the great world of fluid motion. In this section we give some singular perturbation models which arise in some of the above mentioned areas. We will omit the techniques used to solve these models. Interested readers can see the references

cited along with these models, for more details.

(1) Consider the symmetric p-n junction with piecewise constant doping. With an appropriate scaling, the resulting one dimensional stationary problem [211] is given by

$$\begin{cases} \varepsilon\dot{e} = n - p - 1 \\ \varepsilon\dot{p} = -pe - \frac{\varepsilon}{2}J \ , \ \ p(1) = \frac{\delta^2}{n(1)} \\ \varepsilon\dot{n} = ne + \frac{\varepsilon}{2}J \ , \ \ n(0) = p(0) \ , \ \ n(1) = p(1) + 1 \\ \varepsilon\dot{\psi} = e \ , \ \ \psi(0) = 0 \end{cases}$$

on $0 \le x \le 1$, where the small positive parameter $\varepsilon$ represents a scaled Debye lenght (typically about $10^{-\frac{7}{2}}$) and where $J$ and $\delta$ are fixed positive constants representing the current density and an intrinsic carrier concentration. Also, $e$ is the electric field and $p$ and $n$ denotes hole density and electron density, respectively, and both are positive.

(2) Consider the one dimensional Schrödinger equation [57]:

$$\varepsilon^2\frac{d^2\psi_\varepsilon}{dx^2} + (\lambda_\varepsilon - V(x))\psi_\varepsilon = 0 \ , \ \ ||\psi_\varepsilon|| = 1$$

with $V(x)$ (potential) continuous and leading to $+\infty$ as $|x| \to \infty$ and $\varepsilon = h(2m)^{1/2}/(2\pi)$, $h$ denoting Planck's constant and $m$ the mass. Associated to these ODE, the eigenvalue problem in the Hilbert space $L^2(-\infty, +\infty)$ with the norm $||u|| := \left(\int_{-\infty}^{+\infty} u^2(x)dx\right)^{1/2}$ has a discrete spectrum with eigenvalues $\lambda_{\varepsilon,1}, \lambda_{\varepsilon,2}, \cdots, \lambda_{\varepsilon,n}, \cdots$ and eigenfunctions $\psi_{\varepsilon,1}, \psi_{\varepsilon,2}, \cdots,$ $\psi_{\varepsilon,n}, \cdots$.

(3) The following equation [90] represents a time-independent Fokker-Planck equation for a one dimensional dynamical system with state-independent random perturbations:

$$\varepsilon^2\frac{d^2\phi}{dx^2} + b(x)\frac{d\phi}{dx} = 0 \ , \ \ 0 < \varepsilon << 1 \ , \ \ x \in (0,1) \ , \ \ \phi(0,\varepsilon) = A \ , \ \ \phi(1,\varepsilon) = B$$

where $b(x)$ denotes a gradient field. Under the assumptions that $b'(x)$ is strictly negative throughout the interval $[0,1]$ and that $b(\gamma) = 0$ for some $0 < \gamma < 1$, the above problem is a resonant turning point problem.

(4) The following nonlinear system of equations [218] models the action of a bubble-column reactor, which becomes a singular perturbation problem in the limit of infinitely large Péclet numbers:

$$u'(z) = \alpha\beta(z)^{-1}u + \sigma_G x_0 F(x,y),$$

$$\varepsilon_G x''(z) - (2\alpha\varepsilon_G \beta(z)^{-1} + u)\, x' = -\sigma_G(1 - x_0 x)F(x,y),$$
$$\varepsilon_L y''(z) - y' = Dy + \sigma_L \beta(z)(1+\alpha)^{-1} F(x,y),$$

on $0 < z < 1$, where

$$\beta(z) = 1 + \alpha(1 - z)$$

and

$$F(x,y) = (1+\alpha)\mu^{-1}\beta(z)^{-1} y - x$$

Boundary conditions at the ends $z = 0, 1$ of the column are given as

$$u(0) = 1 \ , \ \varepsilon_G x'(0) - x(0) = -1 \ , \ \varepsilon_L y'(0) - y(0) = 0 \ , \ x'(1) = 0 \ \text{and} \ y'(1) = 0$$

In the above, $z$ is a length variable measured along the axis of the column and scaled with respect to the length of that column, $x(z)$ is the reduced mole fraction of the gas component of reactant in the gas phase, while $y(z)$ and $u(z)$ represent the dimensionless concentration of gas in the liquid phase and the superficial gas velocity along the column, respectively, $\varepsilon_G$ and $\varepsilon_L$ are the reciprocal of the gas and liquid phase Péclet numbers, while $\sigma_G$ and $\sigma_L$ are the gas and liquid phase Stanton numbers, respectively, $\mu$ is a constant of proportionality for the rate of absorption of the gas in liquid, $\alpha$ is another constant which denotes a linear pressure field and $D$ is the Damköhler number, which measures the rate of gas uptake in reaction relative to the total liquid convection rate.

(5) The following dimensionless Young-Laplace equation [152] which is a governing equation for measuring the height of the meniscus on a slender vertical cylinder is given by

$$\frac{d^2 Z}{dR^2} = \left\{1 + \varepsilon^2 \left(\frac{dZ}{dR}\right)^2\right\} \left\{Z \left[1 + \varepsilon^2 \left(\frac{dZ}{dR}\right)^2\right]^{1/2} - \frac{1}{R}\frac{dZ}{dR}\right\}$$

$$\frac{dy}{dx} = -tan\phi \ \text{ at } \ x = r_0 \ ; \ \ y \to 0 \ \text{ as } \ x \to \infty$$

where $\varepsilon$ is the perturbation parameter and is given by $\varepsilon = r_0/l_c$ (capillary number),

$Z = y/r_0$, $R = x/l_c = \varepsilon r$, $l_c = (\sigma/\rho g)^{1/2}$, $\rho$ is the density of the liquid, $\sigma$ is the interfacial tension, $g$ is the gravitational acceleration, $r_0$ is the radius of the cylinder, $l_c$ is the capillary length, $\phi$ is an angle between liquid and the solid surface.

(6) In the modelling of a semiconductor device, the model equations [158] governing the static one dimensional case are

$\psi'' = \frac{q}{\varepsilon}(n - p - C(z))$ Poisson's equation

$n' = \frac{\mu_n}{D_n} n\psi' + \frac{1}{qD_n} J_n$ electron current relation

$p' = -\frac{\mu_p}{D_p} p\psi' - \frac{1}{qD_p} J_p$ hole current relation

$J_n' = qR(n,p)$ continuity equation for electron

$J_p' = -qR(n,p)$ continuity equation for holes

for $-l \le z \le l$ subject to the boundary conditions

$\psi(-l) = U_T\, ln\frac{n_i}{p(-1)} + U_A$ (anode) , $\psi(l) = U_T\, ln\frac{n_1}{n(i)} + U_C$ (cathode) , $n(\pm l)p(\pm l) = n_i^2$ and $n(\pm l) - p(\pm l) - C(\pm l) = 0$

where $\psi$ is potential, $J_n$ is electron current density, $J_p$ is hole current density, $n$ is electron density, $p$ is hole density, $q$ is electron charge, $\varepsilon$ is permittivity constant, $\mu_n$ is electron mobility, $\mu_p$ is hole mobility, $D_n$ is electron diffusion constant, $D_p$ is hole diffusion constant, $n_i$ is intrinsic number, $U_T \equiv D_n/\mu_n \equiv D_p/\mu_p$ is thermal voltage, $C(z) = N_D^+(z) - N_A^-(z)$ is impurity distribution, $N_D^+$ is the donor density, $N_A^-$ is the accepter density and $R(n,p)$ is the recombination rate.

(7) Another BVP from semiconductor theory [16] is

$$\varepsilon n' = (n + \beta p)\left[\alpha n\widetilde{f} - \sum_{i=1}^{N_A} \widehat{f}_i - \sum_{j=1}^{N_D} g_j\right] \tag{1.1}$$

$$\varepsilon p' = (n + \beta p)\left[\alpha p\widetilde{f} + \frac{1}{\beta}\sum_{i=1}^{N_A}\widehat{f}_i + \frac{1}{\beta}\sum_{j=1}^{N_D} g_j\right] \tag{1.2}$$

$$n(0) = 1 \;,\; p(1) = 0 \;;\; 0 < x < 1 \tag{1.3}$$

Here $n(x)$ and $p(x)$ are electron and hole densities of negative and positive charges, respectively and $\varepsilon$ is a normalized current density. Values of interest for $\varepsilon$ ranges from 1 to $10^{-12}$. The functions and constants appearing on the right-hand sides of (1.1), (1.2) and (1.3) are given by

$$\widetilde{f} = 1 - n + p - \sum_{i=1}^{N_A} a_i(x)\frac{n + \alpha_i u_i}{n + v_i + \alpha_i(u_i + p)} + \sum_{j=1}^{N_D} d_j(x)\frac{z_j + \delta_j p}{n + z_j + \delta_j(y_j + p)}$$

$$\widehat{f} = \alpha_i A_i a_i(x)\frac{np - v_i u_i}{n + v_i + \alpha_i(u_i + p)} \;,\; g_j = \delta_j D_j d_j(x)\frac{np - y_j z_j}{n + z_j + \delta_j(y_j + p)}$$

$$\beta = 1/3, \quad N_A = 2, \quad N_D = 1, \quad \alpha = 0.05162$$

$$\alpha_i = \delta_j = 1, \quad A_i = D_j = 2.222.10^{-3} \quad \forall\; i, j$$

$$u_1 = 1.854.10^{-4}, \quad u_2 = 0.1021, \quad v_1 == 21.47, \quad v_2 = 3.899.10^{-2}$$

$$y_1 = 2.902.10^{3}, \quad z_1 = 1.371.10^{-6}$$

For small $\varepsilon$, this BVP has a boundary layer at $x = 0$ and an interior (turning point) layer near $x = 1$.

(8) Consider a shock wave in a one dimensional nozzle flow [16]. The steady state Navier-Stokes equations give

$$\varepsilon A(x)uu'' - \left[1 + \frac{\gamma}{2} - \varepsilon A'(x)\right] uu' + u'/u + \frac{A'(x)}{A(x)}\left(1 - \frac{\gamma - 1}{2}u^2\right) = 0, \;\; 0 < x < 1$$

where $x$ is the normalized downstream distance from the throat, $u$ is a normalized velocity, $A(x)$ is the area of the nozzle at $x$, e.g., $A(x) = 1 + x^2$, $\gamma = 1.4$ and $\varepsilon$ is essentially the inverse Reynolds number, e.g., $\varepsilon = 4.792.10^{-8}$. The boundary conditions are

$$u(0) = 0.9129 \text{ (supersonic flow in throat) } , \;\; u(1) = 0.375$$

For this BVP an $O(\sqrt{\varepsilon})$-wide shock develops, whose location depends on $\varepsilon$.

(9) Consider the swirling flow between two rotating, coaxial disks, located at $x = 0$ and at $x = 1$ [16]. The BVP is

$$\varepsilon f'''' + f''' + g' = 0$$

$$\varepsilon g'' + fg' - f'g = 0$$

$$f(0) = f(1) = f'(0) = f'(1) = 0$$

$$g(0) = \Omega_0, \quad g(1) = \Omega_1$$

where $\Omega_0$ and $\Omega_1$ are the angular velocities of the infinite disks, $|\Omega_0| + |\Omega_1| \neq 0$, and $\varepsilon$ is a velocity parameter, $0 < \varepsilon << 1$. For this BVP, multiple solutions are possible. Taking, e.g., $\Omega_1 = 1$, one can obtain different cases for different values of $\Omega_0$. If $\Omega_0 < 0$ (with a special symmetry when $\Omega_0 = -1$) then the disks are counter-rotating; if $\Omega_0 = 0$ then one disks is at rest, while if $\Omega_0 > 0$ then the disks are corotating.

(10) Consider a homogeneous, isotropic, thin spherical shell of constant thickness, subject only to an axisymmetric normal distributed surface load [16]. With $\xi$ the angle between the meridional tangent at a point of the midsurface of the undeformed shell and the base plane, $\phi$ the meridional angle change of the deformed middle surface, $\beta = \xi - \phi$ and $\psi$ a stress function, the following BVP governs the deformation elastostatics of the shell

$$\mu[\psi'' + (cot\xi)\psi' + (v - cot^2\xi)\psi] - \frac{1}{sin\xi}(cos\beta - cos\xi)$$

$$= \mu[vP' + (1+v)Pcot\xi - \frac{1}{sin\xi}(\gamma sin^2\xi)' - v\gamma cos\xi], \quad 0 \leq \xi \leq \pi/2$$

$$\varepsilon^4/\mu[\phi'' + (cot\xi)\phi' + \frac{cos\beta}{sin^2\xi}(sin\beta - sin\xi) - \frac{v}{sin\xi}(cos\beta - cos\xi)] + \frac{sin\beta}{sin\xi}\psi = \frac{cos\beta}{sin\xi}P$$

$$\phi(0) = \psi(0) = \phi(\pi/2) = \psi(\pi/2) = 0$$

Here

$$P(\xi) = -\int_0^\xi (1 - \delta sin\,\eta) cos\,\beta\, sin\,\eta\, d\eta, \quad \gamma = -sin\beta((1 - \delta sin\xi)$$

and $\delta > 1$ a constant, say $\delta = 1.2$ (when a particular load distribution is assumed). Also $v = 0.3$ is a typical value. The parameters $\varepsilon$ and $\mu$ are positive and small (they ralate to the thickness vs radius of the shell). The solution sought has an interior layer in $\phi$ corresponding to a dimpling of the spherical shell. Some representative $(\varepsilon, \mu)$-values for which the problem is fairly difficult are (0.01, 0.0001), (0.001, 0.001), (0.0001, 0.01).

(11) Consider an example from the theory of shells of revolution [16]. The ODEs are

$$\varepsilon^2\left[\psi'' + \frac{1}{x}\psi' - \frac{1}{x^2}\psi\right] - \frac{1}{x}\phi\left(\phi_0 - \frac{1}{2}\phi\right) = 0$$

$$\varepsilon^2\left[\phi''+\frac{1}{x}\phi'-\frac{1}{x^2}\phi\right]+\frac{1}{x}\psi\left(\phi_0-\phi\right)=2\kappa P(x)$$

with $\phi$ and $\psi$ essentially as in the previous example; $\phi_0(x)$ is $\phi$ of the undeformed shell (for a spherical shell $\phi_0(x)=x$) and

$$P(x)=x\left(1-\gamma+\frac{\gamma}{2}x^2\right),\quad \gamma=1.2,\quad v=0.3,\quad \kappa=1$$

The boundary conditions are

$$\phi(0)=\psi(0)=0,\quad \phi(1)=\psi'(1)-v\psi(1)=0$$

For this BVP an interior layer (corresponding to dimpling) forms in a solution for $\phi$. There is an additional boundary layer at $x=1$ and more than one solution exist. The value $\varepsilon=10^{-4}$ (which gives a rather thin shell) yields a challenging numerical problem.

(12) The governing equations in the Reissner-Mindlin plate theory [17] can be written in the form

$$\frac{1}{2}\left[(1-\mu)\,\nabla^2\,\psi-(1+\mu)\,\nabla\,(\nabla.\psi)\right]-\varepsilon^{-1}(\psi+\nabla w)=0$$

$$\varepsilon^{-1}(\nabla^2 w+\nabla.\psi)=\frac{\rho}{D}$$

where $w$ is the the transverse deflection, $\psi=(\psi_x,\psi_y)$ is the vector whose components are the plate rotations in the direction of the coordinate axes, $\mu$ is Poisson's ratio, $\rho$ is the transverse load, $D$ is the plate modulus and $\varepsilon$ is a parameter defined by $\varepsilon=h_0^2/[6(1-\mu)\kappa^2]$, where $h_0$ is the plate thickness and $\kappa^2$ is a constant.

(13) Consider the two-dimensional motion of an incompressible viscous fluid [177] in the $(x,z)$ coordinate system. Assuming that the fluid is stratified in the $z$-direction, i.e., the stationary density $\rho_0$ and the kinematic viscosity $\gamma$ are the functions of the verticle coordinate $z$ only. Suppose gravity is acting in the negative $z$-direction. The linearized Navier-Stokes equations which described the small motion of such fluid then take the form

$$\rho_0\left(\frac{\partial V}{\partial t}-\gamma\Delta V\right)+\nabla p+e_z\rho_1 g=f_1$$

$$\frac{\partial\rho_1}{\partial t}=\rho_0\omega_0^2(e_z,V)/g\ ,\quad div\,V=0$$

where $V=\{v_1,v_2\}$ is the velocity, $(.,.)$ denotes the Euclidean scalar product, $\rho_1$ is the dynamic perturbation of the density, $p$ is the dynamic pressure, $e_z$ is a unit vector in the

$z$-direction, $g$ is the acceleration of gravity, $f_1$ is an external force, $\omega_0$ is the Vaisala-Brunt frequency and $\omega_0^2 = -g(\partial\rho_0/\partial z)/\rho_0(z)$.

(14) The mathematical model describing the motion of the sunflower is the equation [192]:

$$\varepsilon x''(t) + a\,x'(t) + b\,sin\,x(t-\varepsilon) = 0 \ , \ \varepsilon > 0 \ , \ t \in [-\varepsilon, 0]$$

with $x'(0)$ prescribed. Here the function $x(t)$ is angle of the plant with the verticle, the time lag say $\varepsilon$ is geotropic reaction, and $a$ and $b$ are positive parameters which can be obtained experimentally.

(15) Consider the boundary layer flow of an electrically conducting incompressible fluid [245] (with electric conductivity $\sigma$) over a continuously moving flat surface with $B_0$ an imposed, uniform magnetic field perpendicular to the surface. The boundary-layer equation for the flow field (in nondimensionalized form) is then given by

$$\varepsilon\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \frac{\partial u}{\partial t} + Mu \quad \text{for large } R_0 \ , \ (\varepsilon = 1/R_0)$$

$$\frac{\partial^2 u}{\partial y^2} + \varepsilon\frac{\partial u}{\partial y} = \varepsilon\frac{\partial u}{\partial t} + \varepsilon Mu \quad \text{for small } R_0$$

$$u(0,T) = 1 \ , \ u(y,t) \to 0 \text{ as } y \to \infty \ , \ t > 0 \ ; \ u(y,0) = 0, 0 < y < \infty$$

where $R_0 = V_0L/v$, the suction Reynolds number, $M = \sigma B_0^2 L/\rho V_0$, the Hartmann number, $u = U/U_w$, $y = Y/L$, $t = \tau V_0/L$, $L$ is the characteristic length (between the slit and the wind-up roll), $V_0$ is the suction velocity at the plate, $\tau$ is the time and the $X$ and $Y$ axes are taken along and perpendicular to the sheet, $u$ is the velocity field.

(16) Consider the free motion of the undamped linear spring mass system with a very resistant spring. Let the prescribed specific displacement be at times $t = 0$ and 1. Then one can obtain [185] the two-point problem

$$\varepsilon^2\ddot{x} + x = 0, \quad 0 \le t \le 1, \quad x(0) = 0, \quad x(1) = 1$$

where $\varepsilon^2$ (the ratio of the mass to the spring constant) is small. For nonexceptional small positive values of $\varepsilon$ the exact solution oscillates rapidly, so no pointwise limit exists as $\varepsilon \to 0$.

(17) Consider the Dirichlet problem [182], [268]:

$$\varepsilon\ddot{x} + x\dot{x} = o \quad \text{on} \quad 0 \le t \le 1$$

where $x(0)$ and $x(1)$ are prescribed. It could describe the motion of a mass moving in a medium with damping proportional to the displacement, with either the mass small or the damping large. Depending on the particular end values $x(0)$ and $x(1)$, the solution may have initial/shock/boundary layers.

(18) The example [215]:

$$\varepsilon \ddot{x} - \left(t - \frac{1}{2}\right) \dot{x} = 0, \quad 0 \leq t \leq 1, \quad x(0) \text{ and } x(1) \quad \text{are prescribed}$$

relates to an exit time problem for randomly perturbed dynamical systems.

(19) Consider an isothermal atmosphere [8] which is viscous and thermally conducting, occupies the upper half-space $z > 0$. Consider small oscillations about equilibrium which depend only on the time $t$ and on the verticle coordinate $z$. Let $p$, $\rho$, $w$ and $T$ be the perturbations in the pressure, density, vertical velocity and temperature, and $P_0$, $\rho_0$ and $T_0$ are the equilibrium quantities. Then the linearized equations of motion are

$$\rho_0 w_t + p_z + g\rho = 4\mu w_{zz}/3$$

$$\rho_t + (\rho_0 w)_z = 0$$

$$\rho_0[c_V(T_t + qT) + gHw_z] = \kappa T_{zz}$$

$$P = R(\rho_0 T + T_0 \rho) \quad \text{with prescribed boundary conditions}$$

where, $\mu$ is the dynamic velocity coefficient, $\kappa$ is the thermal conductivity, $c_V$ is the specific heat at constant volume and $q$ is the Newtonian cooling which refers to the heat exchange and proportional to the temperature perturbation associated with the wave, all assumed to be constant, $R$ is the gas constant, $g$ is the gravitational acceleration and $H = RT_0/g$ is the density scale height.

Some models on optimal control can be found in Ardema [11] and in Kokotovic [136] and the references cited in [136]. Moreover other good models can be found in Kellogg [133], Bassanini and Elcrat [22], Szmolyan [242], Amick and McLeod [10], Giovangigli [88], Sobolev [220], Wolfe [270] and the references cited therein. Few more models can be found in the book by Miller [166] which contains a collection of papers dealing with singular perturbation problems that arise in various areas of science and engineering.

## 1.3 Numerical Treatment of Singular Perturbations Problems

Starting from the Pearson's work [189], [190] in the year 1968 and upto the year 1984, the numerical methods for one dimensional singular perturbation problems have been surveyed in the first author's paper [120]. Here we present the survey of the work done after the year 1984 in this area and in a chronological order as the field developed year after year from 1984. Due to space limitations we include the work on one dimensional problems only.

Ascher [15] considered singularly perturbed boundary value ordinary differential problems. He used two families of $A$ - stable symmetric schemes, namely, the $K$ - stage Gauss and the $K+1$ - stage Lobatto Runge-Kutta methods of order $2K$. It is shown that for linear BVPs which have only fast solution components, or slow solution components but with a weak coupling between fast and slow, that these two classes of methods have similar stability properties although the Gauss methods lose their superconvergence while Lobatto methods do not. However, for problems in which both fast and slow components are present with no restriction on the coupling, the Lobatto methods are less stable although still retaining superconvergence. This seems to be due to the fact that the Gauss methods are algebraically stable while the Lobatto methods are not, thus indicating that while Lobatto methods are more accurate for some classes of problems, Gauss methods are in general, more robust.

Markowich [159] Discussed a special-purpose finite-difference scheme for the one dimensional stationary semiconductor device equations. They consist of three second order differential equations concerning potential distribution, carrier concentration and current flow with given boundary conditions. Because the second derivative of the potential equation is multiplied by a small parameter(typically about $10^{-8}$) one is led to a singularly perturbed problem. Assuming that the semiconductor device has only one junction and that the doping profile has jump discontinuities at the junction, he first proved the existence of internal layers corresponding to these discontinuities. Then a finite difference scheme is given for the resolution of the internal layers with the property of requiring a reasonable number of grid points. The relation of this scheme to exponentially fitted

schemes is discussed as well as its convergence and the construction of efficient grids.

Earlier to this, Markowich and Ringhofer [158] derived a model for a pn-junction with an applied bias. They treat the static one dimensional case which yields a singular perturbation problem on [-1,1]. Assuming the doping profile as an odd function, they restricted the problem to [0,1] and considered as a problem of boundary layer type. By neglecting hole and electron currents, they modeled the equilibrium case(zero applied bias). The resulting second-order singular perturbation problem is solved numerically by spline collocation and compared with the nonequilibrium case. The approximate model in the equilibrium case is seen to be reasonable for small forward or moderate reverse bias. For the nonequilibrium case, an asymptotic expansion is used. The reduced system and the boundary layer systems are formed. It is shown that a layer only occurs at 0. Existence and qualitative behaviour are obtained for both the boundary layer equation and the reduced equation. Using the asymptotic expansion as initial guesses, the singular perturbation problems are solved numerically and the dependence of solutions on the applied bias is analysed in various cases.

Enright [72] reported on an ongoing investigation into the performance of numerical methods for two-point boundary value problems. He outlined how methods based on multiple shooting, collocation and other local discretizations can share a common structure. The identification of this common structure permits one to analyse how various components of a method interact and also permits one to consider the assembly of a collection of modular routines which will eventually form the basis for a software environment for solving two-point boundary value problems. He implemented and analysed a family of multiple shooting methods as well as a family of collocation/Runge-Kutta methods. He investigated the numerical conditioning of both families of methods on such problems and the convergence requirements of the corresponding iteration schemes.

Surla and Jerkovic [235] considered the singularly perturbed boundary value problem:

$$-\varepsilon u''+a(x)u'+b(x)u=f(x),\quad x\in[0,1]\ ;\ \alpha_1u(0)+\alpha_2u'(0)=\gamma_0,\quad \alpha_3u(1)+\alpha_4u'(1)=\gamma_1$$

where $b(x)\geq b>0$, $|\alpha_1|+|\alpha_2|\neq 0$, $|\alpha_3|+|\alpha_4|\neq 0$, $\alpha_1\cdot\alpha_2\leq 0$, $\alpha_3\cdot\alpha_4\geq 0$. Using spline collocation method, they demonstrated that the exponential behaviour of the exact solution is transferred directly to the spline coefficients by the fitting factor. Thus, a

suitable choice of the fitting factor guarantees uniform stability of the system matrix, while the order of convergence is preserved for fixed $\varepsilon$.

Niijima [181] analysed a nonmonotone difference scheme for solving a nonlinear singular perturbation problem

$$\varepsilon y'' - (f(y))' - b(x,y) = 0 \ , \ 0 < x < 1 \ ; \ y(0) = A \ , \ y(1) = B$$

with small positive parameter $\varepsilon$. She proved existence and uniqueness of approximate solutions and convergence in $L^1$ norm. The uniform bounded variation estimate for approximate solutions is essential for the $L^1$ convergence result and was proved for monotone difference schemes by Abrahamsson and Osher [3]. This property is established for a nonmonotone scheme in the Niijima's above paper. Some numerical computations are also presented.

Maier [156] gave a collocation method for the numerical solution of a singularly perturbed boundary value problem of the form:

$$\left.\begin{array}{ll} y' = f(x,y,\varepsilon) & \\ r(y(a),y(b)) = 0, & a \le x \le b \\ y = (y_1,\cdots,y_n)^t & y_i : [a,b] \to R \\ f = (f_1,\cdots,f_n)^t & f_i : [a,b] \to R \\ r = (r_1,\cdots,r_n)^t & r_i : R \times R \to R \end{array}\right\} \tag{1.4}$$

Based on an algorithm of Dickmanns and Well [59], the solution of (1.4) is approximated by piecewise cubic Hermite polynomial functions. Inside the layers, the cubic polynomials are replaced by tension splines. The tension parameters are selected from the eigenvalues of the functional matrix of (1.4). Inside the boundary layers the solution is determined by eigenvalues with absolutely large real parts and therefore one of these eigenvalues is chosen as tension parameter. In the outer region tension parameter is taken as zero which leads to cubic polynomials in this region. He solved three problems arising in the physical theory of semiconductor devices.

O'Malley [183] provided a broad survey concerning boundary value problems for certain systems of nonlinear singularly perturbed ODEs. He emphasized important and difficult open problems needing much more study, in terms of both mathematical and numerical analysis point of view and with computational experiments supporting the theory.

Till 1985, the development of general numerical methods for singular perturbation problems whose solution exhibit internal layer type behaviour has been largely neglected. Indeed, even the analytic study of general systems of first order ODEs with this type of behaviour appears to be quite limited; perhaps this is one of the reason for the lack of progress in this area. The theory for difference approximations for such problems with turning points, is less well developed (till 1985). One would like to have a result of the type that if a mesh has been constructed which resolve the features(e.g., internal and boundary layers) of the solution of the analytic problem, then the difference approximation that one applies on this mesh will give an accurate solution. Regardless of which difference methods are applied, the main practical difficulty for turning point problems is one of constructing an appropriate mesh on which to solve the problem numerically. Keeping above in mind, Brown [36] discussed how to determine a priori that where the solutions of a system of ODEs can be expected to vary rapidly. He applied the one-sided difference approximations by constructing a change of variables which separates the rapidly growing, rapidly decaying and slowly varying components of the solution.

Sakai and Usmani [208] gave a new concept of $B$-spline bases for hyperbolic and trigonometric splines which are different from earlier known ones. It is proved that the hyperbolic and trigonometric $B$-splines are characterized by a convolution of some special exponential functions and a characteristic function on the interval $[0,1]$. On this basis one obtains properties of these hyperbolic and trigonometric $B$-splines similar to those of the polynomial ones. An application of a hyperbolic spline of degree 4 is given to a numerical solution of the following simple singular perturbation problem:

$$\varepsilon y''(x) - y(x) = \alpha(x) \ , \ 0 \le x \le 1 \ ; \ y(0) = \alpha \ , \ y(1) = \beta \ ; \ 0 < \varepsilon \ll 1$$

Based on the method of inner boundary condition for solving singular perturbation problems, Kadalbajoo and Reddy [112] gave a numerical method for the singular perturbation problem

$$\varepsilon y''(x) + f(x)y'(x) + g(x)y(x) = h(x) \ , \ 0 \le x \le 1 \ ; \ y(0) = A \ , \ y(1) = B$$

where $\varepsilon$ is a small positive parameter, $A$, $B$ are given constants, $f(x) \ge M > 0$ ($M$ is constant), and $g(x)$ and $h(x)$ are assumed to be sufficiently continuously differentiable

functions in $[0,1]$. The problem is partitioned into inner and outer region problems. They took the outer solution in the following form

$$y(x)=\sum_{n=0}^{\infty}a_n(x)\varepsilon^n \ , \ \ x_p\le x\le 1 \ , \ \ y(1)=B$$

where $x_p$, $0<x_p<1$, is the terminal point. Using $Y(x_p)=C=y(x_p)$ and $Y(0)=A$, one obtains a new inner region problem with the solution $Y(x)$. This problem is solved as a two-point boundary value problem and provides the terminal condition $y'(x_p)=D=Y'(x_p)$ for the outer region problem. The new outer region problem with boundary conditions $y'(x_p)=D$, $y(1)=B$ is solved by employing the classical finite difference scheme. The method is iterative on the terminal point $x_p$. The process is to be repeated for various choices of $x_p$ until the solution profiles stabilize in both regions.

Using a framework inspired by theoretical multiple shooting, Dieci and Russell [60] considered a general class of numerical methods for solving boundary value problems for ordinary differential equations. They introduced an extension of the concept of dichotomy which provides a natural way to analyze this performance. On one hand their framework helps to explain the ineffectiveness of some methods. On the other, it leads one to reconsider old ideas in the search for new ways to handle these difficult problems. The renewed interest in these methods is, in fact, partly due to the theoretical justification gained when using this framework for problems having boundary and interior layers.

Kadalbajoo and Reddy [113] considered linear singular perturbation problems in the form of two-point boundary value problems as:

$$\varepsilon y''(x)+a(x)y'(x)-b(x)y(x)=f(x) \ , \ \ 0\le x\le 1 \ ; \ \ y(0)=\alpha \ , \ \ y(1)=\beta$$

Here, $\varepsilon$ is assumed to be a very small positive number and the various functions are assumed to have appropriate smoothness properties and to be bounded. The basic idea is to introduce a small deviating or retarded argument $\delta$ and to expand the second-order term in the BVP. This converts the problem into a first-order functional-differential equation. This equation is then integrated by parts, with the trapezoidal rule. More expansions and manipulation lead to a three-term recurrence relation. Discrete invariant imbedding algorithm is used to solve a tridiagonal algebraic system. Stability properties are given and a criterion is given for picking $\delta$ a posteriori. Some computational results are also given.

Vulanović [256] solved the singularly perturbed problem

$$-\varepsilon u'' - b(x)u' + c(x)u = f(x)$$

subject to one of the following boundary conditions: $u(0) = \gamma_0$, $u(1) = \gamma_1$, or $-\varepsilon u'(0) = \gamma_0$, $u(1) = \gamma_1$. The functions $b$, $c$, $f$ are sufficiently smooth and $b(x) > \beta > 0$, $c(x) \geq 0$, while $0 < \varepsilon \ll 1$. He obtained the second-order convergence uniform in $\varepsilon$ due to the treatment of the boundary layer function, to a special nonequidistant mesh which is dense in the layer, and to the use of a combination of central and mid-point finite-difference schemes.

Berger [28] considered the singularly perturbed problem

$$\varepsilon u_{xx} + (b(x)u)_x = f(x) \quad , \quad 0 < x < 1$$

where $u(0)$ and $u(1)$ are given, $b > 0$, and $\varepsilon$ is a parameter in $(0, 1]$. He obtained the bounds on $u$ and its derivatives. A conservative three-point difference scheme for the problem is analyzed on a uniform mesh of width $h$. It is shown that the nodal errors are bounded by $C(h^2 + \varepsilon)$ when $\varepsilon \leq h$ and by $Ch^2/\varepsilon$ when $h \leq \varepsilon$, where $C$ is a constant independent of $h$ and $\varepsilon$.

Stynes and O'Riordan [229] applied a Petrov-Galerkin finite element method with exponential basis elements to a nonselfadjoint singularly perturbed two-point boundary value problem in conservative form:

$$\varepsilon(pu')' + (qu)' = f \;\text{ on }\; (0,1) \;\; ; \;\; u(0) = A \, , \; u(1) = B$$

where, $0 < \varepsilon \leq 1$; $A$ and $B$ are given constants, the coefficients $p$, $q$ and $f$ are in $C^2[0,1]$ and satisfy $p(x) \geq \alpha > 0$, $q(x) \geq \beta > 0$. This method is shown uniformly first order accurate in $L_\infty$ and uniformly second order accurate at the nodes.

Again Stynes and O'Riordan [230] examined the problem

$$\varepsilon u'' + a(x)u' - b(x)u = f(x)$$

for $0 < x < 1$, $a(x) \geq \alpha > 0$, $b(x) \geq \beta$, $\alpha^2 + 4\alpha\beta > 0$; $a$, $b$ and $f$ in $C^2[0,1]$, $\varepsilon$ in $(0, 1]$, $u(0)$ and $u(1)$ given. Using finite elements and a discretized Green's function, they showed that the El-Mistikawy and Werle difference scheme on an equidistant mesh of

width $h$ is uniformly second order accurate for this problem. With a natural choice of trial functions, they obtained uniform first order accuracy in $L^\infty(0,1)$ norm. Choosing piecewise linear trial functions("hat" functions) they obtained the same accuracy in the $L^1(0,1)$ norm.

O'Riordan and Stynes [186] considered the following selfadjoint singular perturbation problem:

$$\varepsilon^2(py')' - ry = f \text{ on } (0,1) \ ; \ y(0) = A \ , \ y(1) = B$$

where, $\varepsilon$ is a parameter in $(0,1]$, $A$ and $B$ are given constants; $p, r, f \in C^2[0,1]$ and $p(x) \geq \alpha > 0$, $r(x) \geq \beta > 0$ for $x \in [0,1]$. They applied finite element method with exponential basis elements to this problem. The tridiagonal difference scheme generated is shown to be uniformly second-order accurate for this problem. With a certain choice of trial functions, they obtained uniform first-order accuracy in $L^\infty[0,1]$ norm.

In 1984, Boglaev and Stanilovskii [31] published a comprehensive bibliography of numerical techniques for singularly perturbed linear and nonlinear boundary value problems. Ignatev and Zadorin's paper [106] supplements this bibliography by a comparative study of numerical methods for nonlinear problems of the type

$$\varepsilon u'' + a(u,x)u' = f(u,x) \ ; \ u(0) = A \ , \ u(1) = B \ ; \ \varepsilon > 0$$

Under the hypothesis $a(u,x) \geq \alpha > 0$, they consider (i) combinations of initial and boundary value techniques, (ii) modifications of the generalized Newton-Raphson method, (iii) Galerkin-type methods, (iv) finite difference schemes and (v) a method of directed differences. Estimates for the absolute error of the computed solution are given. They studied the behaviour of the exact solution as well as the use of finite difference schemes for special cases of the above problem. Finally, they considered boundary value problems of the form

$$\varepsilon u'' + f(x,u,u',\varepsilon) = 0 \ , \ u(0) = A \ , \ u(1) = B \ ; \ \varepsilon > 0$$

Farrell [76] gave some results which characterize the behaviour of a linear nonselfadjoint singular perturbation problem. He also gave criteria for uniform convergence of a nonturning, simple turning point and one multiple turning point case and indicate

the uniform methods for higher order cases. Then he discussed the consequences for quasilinear problems.

In [77] Farrell considered some difference schemes for the solution of the following model differential equation which is a singular perturbation problem without turning points:

$$\varepsilon u_\varepsilon''(x) + a(x)u_\varepsilon'(x) - b(x)u_\varepsilon(x) = f(x), \quad (0 < x < 1) \quad ; \quad u_\varepsilon(0) = A \, , \; u_\varepsilon(1) = B$$

where, $a, b, f \in C^k[0,1], k \geq 2$ and $a(x) \geq \underline{a} > 0$. He defined a stronger convergence criterion known as uniform convergence, which causes the difference scheme to represent the rapidly varying components of the solution accurately. For the proposed class of difference schemes, he showed the necessary and sufficient conditions for uniform convergence. By solving one numerical example, he presented and compared the experimental and theoretical rates of convergence for a number of well known schemes, e.g., Samarskii [75], Abrahamsson-Keller-Kreiss [1], Il'in fitting ([9], [107]), constant Il'in [40], El-Mistikawy-Werle [71] and few others.

O'Riordan and Stynes [187] considered the numerical solutions of the differential equation

$$\varepsilon(p(x)u')' + (q(x)u)' - r(x)u = f(x) \;\; , \;\; 0 < x < 1 \;\; ; \;\; u(0) = u_0 \;\; , \;\; u(1) = u_1$$

where $p > 0$, $q > 0$, $r \geq 0$, $0 < \varepsilon \leq 1$, and $p$, $q$, $r$ and $f \in C^2[0,1]$. Using finite elements with uniform mesh $h$, they generate a tridiagonal difference scheme which has uniform $O(h^2)$ nodal accuracy. Using piecewise linear trial functions, they obtain uniform $O(h)$ accuracy in the $L^1(0,1)$ norm. Using certain other trial functions ($\overline{L}$-splines), they obtain uniform $O(h)$ accuracy in the $L^\infty(0,1)$ norm.

Stynes and O'Riordan [231] gave a nonlinear difference scheme for the following semilinear singularly perturbed two-point boundary value problem:

$$\varepsilon y''(x) + a(x)y' - d(x,y) = 0 \quad ; \quad y(0) = A \, , \; y(1) = B$$

where $A$, $B$ are given constants, $\varepsilon$ is a small parameter in $(0,1]$, $a(x) \in C^1[0,1]$, $d \in C^1([0,1]\times R)$, $d_y(x,y) \geq \delta > 0$ on $[0,1]\times R$ and $d_y(x,y) + a'(x) \geq \delta > 0$ on $[0,1]\times R$, where $\delta$ is independent of $x$ and $y$. Using this scheme the solution is shown to be first-order

accurate in the discrete $L^1$ norm, uniformly in the perturbation parameter. Excluding the turning points, the scheme is first-order accurate in the discrete $L^\infty$ norm.

Boglaev [32] considered the numerical solution of a vector singularly perturbed problem of the form

$$\mu \frac{dz}{dt} = F(z,t) \qquad z(0) = z_0$$

where $\mu$ is a small positive parameter and $z$ and $F(z,t)$ are vector functions. In constructing a numerical method for the solution of this problem he used the method (employed by many investigators) of extracting the linear part with respect to $z$, which he called extraction of the principal part of the differential operator of the problem indicated. Using this scheme, he constructed explicit and implicit Euler methods, which he calls explicit and implicit one-step methods, as well as a centralized scheme with degree $p = 2$. He proved the uniform convergence of these methods with respect to the small parameter. For the implementation of the implicit schemes he proposed an iterative method and proved the convergence of these schemes.

In the year 1987, a series of articles by Kadalbajoo and Reddy appeared (see, e.g., [114], [115], [116], [118]) for the numerical solution of the following class of singular perturbation problems:

$$\varepsilon y''(x) + a(x)y' - b(x)y = f(x) \ , \ \ 0 \le x \le 1 \ ; \ \ y(0) = \alpha \ , \ y(1) = \beta$$

where $\varepsilon$ is a small positive parameter; $\alpha$, $\beta$ are given constants; $a(x)$, $b(x)$ and $f(x)$ are assumed to be sufficiently continuously differential functions in [0,1], and $b(x) \ge 0$, $a(x) \ge M > 0$ on [0,1], where $M$ is some positive constant. Under these assumptions, the above problem has a unique solution $y(x)$ which, in general, displays a boundary layer of width $O(\varepsilon)$ at $x = 0$ for small values of $\varepsilon$. In [114], [115] and [116], the original problem is divided into inner and outer region problems. In [114] the reduced problem is solved to obtain the terminal boundary condition. Then a new inner region problem is created and solved as a two point boundary value problem. In turn, the outer region problem is also modified and the resulting problem is efficiently treated by employing the trapezoidal formula coupled with a discrete invariant imbedding algorithm. The proposed method was iterative on the terminal point. In [115] a terminal boundary condition in implicit form is derived. Then the outer region problem is solved as a two-point boundary

value problem, and an explicit terminal boundary condition is obtained. This leads to a new inner region problem which is solved as a two point boundary value problem using the explicit terminal boundary condition. This method was iterative on the terminal point of the inner region. In [115] they obtained the terminal boundary condition by introducing a small deviating argument whereas in [116] it is obtained directly through the reduced equation (by putting $\varepsilon = 0$ in the original singular perturbation problem) at that terminal point. In [118] the original second order differential equation is replaced by an approximate first order differential equation with a small deviating argument and is solved efficiently by employing the Simpson rule coupled with the discrete invariant imbedding algorithm. This method is iterative on the deviating argument. In [117] they considered the following class of nonlinear singular perturbation problems:

$$\varepsilon y''(x) + [p(y)y(x)]' + q(x, y(x)) = r(x) \ , \ \ a \le x \le b \ ; \ \ y(a) = \alpha \ , \ y(b) = \beta$$

$0 < \varepsilon << 1$; $\alpha$, $\beta$ given constants. Under the appropriate assumptions on $p$, $q$ and $r$ the above problem has a unique solution which displays a boundary layer of width $O(\varepsilon)$ at $x = a$ for small values of $\varepsilon$. The original second order problem is replaced by an asymptotically equivalent first-order problem and is solved as an initial value problem. In all the above papers ([114], [115], [116], [117] and [118]) numerical examples are solved.

Using a finite difference framework of Lynch and Rice [155] and Doedel [64], Gartland [83] constructed a family of uniformly accurate finite-difference schemes for the model problem:

$$-\varepsilon u''(x) + a(x)u' + b(x)u = f(x) \ , \ \ 0 < x < 1 \ ; \ \ u(0) = g_0 \ , \ u(1) = g_1$$

with the assumptions that $a$, $b$ and $f$ are bounded continuous functions and $a(x) \ge \underline{a} > 0$ on [0,1]. A scheme of order $h^p$ (uniform in $\varepsilon$) is constructed to be exact on a collocation of functions of the type

$$\left\{1, x, \cdots\cdots, x^p, exp\left(\int_x^1 a\right), x\, exp\left(\int_x^1 a\right), \cdots\cdots, x^{p-1}\, exp\left(\int_x^1 a\right)\right\}$$

The high order is achieved by using extra evaluations of the source term $f$. The details of the construction of such a scheme(for general $p$) and a complete discretization error analysis, which used the stability results of Niederdrenk and Yserentant [178] are given in

this paper. He also presented some numerical experiments which exhibit uniform orders $h^p$, $p = 1, 2, 3$ and 4.

Lin and Jiang [146] studied a singularly perturbed second-order equation with periodic boundary conditions:

$$\varepsilon u''(x) + a(x)u' - b(x) = f(x) \ , \ x \in (0,1) \ ; \ u(0) = u(1) \ , \ u'(1) - u'(0) = C\varepsilon^{-1}$$

where $\varepsilon$ is a small positive parameter, $a(x)$, $b(x)$, $f(x)$ are sufficiently smooth functions with period 1 and $C$ is a given constant. Previously in [191], Pechenkina developed an $N$-step difference scheme for this problem and showed that this scheme converges uniformly on $[0,1]$ with $O(N^{-1/2})$. In [146], Lin and Jiang use the method of decomposing the singular term from its solution and combining an asymptotic expansion of the equation to prove that Pechenkina's scheme actually converges uniformly with $O(N^{-1})$.

O'Malley [184] presented several different kinds of singular perturbation problems arising from ordinary differential equations when a small parameter multiplies a highest derivative. The paper begins with initial value problems of first and second order, using relatively simple examples to explain the approach. Typically, setting $\varepsilon = 0$ in the differential equation yields a "reduced" differential equation whose solution must be matched with solutions describing the behavior in layers. Several sketches are included that enable the reader to grasp the main points with ease. The paper then moves on to so-called "singular" singular perturbation problems. These are systems of the form

$$\varepsilon \frac{dy}{dt} = g(y, t, \varepsilon)$$

together with initial conditions. For these, the reduced problem is typically not a differential equation, but is instead a relationship among the dependent variables—the equilibrium manifold. The initial conditions do not in general lie on this manifold, and so a "dynamic" manifold is introduced to accommodate the rapid transition from the initial conditions to the equilibrium manifold. Then singularly perturbed boundary value problems for scalar equations are discussed. The approach is to solve the reduced equation to obtain the behavior of the exact solution everywhere except in narrow regions at the boundaries or in the interior, the solution in these later regions being described by equations obtained after the independent variable has been stretched. Again, the sketches are

very helpful. In last section he treated singularly perturbed boundary value problems, but emphasized phase plane methods. Once again, the explanations are very clear, and are amply illustrated by specific examples and accompanying sketches. A discussion of phase plane methods can also be found in his book *Singular perturbation methods for ordinary differential equations* [185].

Dieci et al. [62] discussed the implementation aspects of a Riccati transformation method for solving linear boundary value problems with separated end conditions which was analysed in their previous paper [61]. The resulting algorithm is based on a double sweep (bidirectional) integration process: $q+1$ integrations from left to right and 1 integration from right to left. Although this approach requires fewer integrations than all the other implementations known hitherto, continuous representations of the forward integrated part of the solution are needed in order to be able to perform the return integration. Once this is done, however, the construction of a continuous representation of the solution itself becomes an easy matter. The efficiency of the method is demonstrated on a number of singular perturbation problems. Particular attention is paid to the choice of scaling for the variables and its effect on the solution method. For one fascinating example

$$\varepsilon y'' - ty' - y/2 = 0 \ , \quad 0 < t < 1 \ ; \ y(-1) = 1 \ , \quad y(1) = 2$$

it can be seen that the phenomenon of superconvergence, as investigated by Dahlquist et al. [53], can be a potential difficulty in the integration of the Riccati differential equation. The two parts of this paper are good source of information for those who would like to learn more about the use of initial value techniques in the numerical treatment of two-point boundary value problems.

Gartland [84] considered a linear singularly perturbed two-point boundary value problem without turning points. Under minimal hypotheses on the coefficients in the differential equation, he gave an elegant proof that the El Mistikawy and Werle discretization on a regular mesh is second-order accurate, uniformly in the perturbation parameter. Also under slightly weaker assumptions for an equivalent Petrov-Galerkin formulation, he proved the global uniform $O(h)$ convergence.

Surla and Stojanovic [238] solved the selfadjoint singularly perturbed boundary prob-

lem:

$$Ly = -\varepsilon y'' + p(x)y = f(x) \ , \ p(x) \geq p > 0 \ , \ 0 < x < 1$$

$$y(0) = \alpha_0 \ , \ y(1) = \alpha_1 \ ; \ \alpha_0 \ , \ \alpha_1 \in R$$

A solution technique based on tension splines, which was used in the case of a nonselfadjoint problem by M. K. Jain and T. Aziz [111], is applied to the above problem. For $p(x) = p$ =const and $f(x) \in C^2[0,1]$, Surla and Stojanovic showed the $O(h^2)$ uniform convergence of the related difference scheme between the grid points, while at the grid points, the convergence is shown to be $O(h \min(h, \sqrt{\varepsilon}))$, provided the condition on $p(x)$ is $p(x) \geq p > 0$, $p'(0) = p'(1) = 0$.

Vulanović [256] constructed higher order monotone finite difference schemes on the uniform mesh for the following nonlinear singularly perturbed problem

$$-\varepsilon u'' - b(u)u' + c(x,u) = 0, \ x \in [0,1]; \ u(0) = U_0, \ u(1) = U_1$$

where $0 < \varepsilon \ll 1$, the functions $b$ and $c$ are sufficiently smooth and $b(u) \geq b_* > 0$ for $u \in R$, $C_u(x,u) \geq 0$ for $x \in [0,1]$ and $u \in R$. $U_0$ and $U_1$ are given numbers. He showed that there is no need for the standard decomposition of the discretization matrix which was used by Roos [202].

Ibrahim and Temsah [105] gave three spectral methods with interface points for the numerical solution of the following singular perturbation problems:
(1) The stiff initial-value problems

$$\varepsilon u'(x) + q(x)u(x) = f(x), \ a \leq x \leq b, \ u(a) = \alpha$$

(2) Singularly perturbed two-point boundary value problem

$$\varepsilon u''(x) + p(x)u'(x) + q(x)u(x) = f(x), \ a \leq x \leq b, \ u(a) = \alpha \ , \ u(b) = \beta$$

They referred these methods as collocation method I, Collocation method II and Spectral tau method. The collocation method I employs the pseudospectral(collocation) approximation and generates approximations to the higher order derivatives of the function through successive differentiation of the Chebyshev polynomial approximation to the approximate solution. The collocation method II employs the pseudospectral(collocation)

approximation and generates approximations to the lower order derivatives of the function through successive integrations of the Chebyshev polynomial approximation to the highest order derivative. Spectral tau method makes use of global orthogonal test and trial functions. Good numerical results have been obtained and compared with other methods.

Liseiken and Petrenko [150] considered the numerical approximation of several singularly perturbed ordinary differential systems of the form:

(i) $(\varepsilon + px)^b u'' + a(x)u' - f(x,u) = 0,\ 0 < x < 1,\ 0 < \varepsilon \leq 1,$

(ii) $u(0) = l_0,\ u(1) = l_1$, with $a(x)$ and $f(x,u)$ continuous.

They considered three types of equation:

(a) $p = 0,\ b = 1,\ f_u(x,u) \geq c > 0,$

(b) $p = 0,\ b = 1,\ f_u(x,u) = xg_u(x,u) \geq 0,\ g(x,u)$ continuous,

(c) $p = 1,\ b$ a positive integer and $f_u(x,u) \geq 0$.

The system (i), (ii) is replaced by the difference scheme

$$(i)' \quad \frac{2(\varepsilon + px_i)^b}{h_i + h_{i-1}} \left( \frac{u_{i+1} - u_i}{h_i} - \frac{u_i - u_{i-1}}{h_{i-1}} \right)$$
$$+ a_+(x_i) \left( \frac{u_{i+1} - u_i}{h_i} \right) + a_-(x_i) \left( \frac{u_i - u_{i-1}}{h_{i-1}} \right) - f(x_i, u_i) = 0$$

$$(ii)' \quad u_0 = l_0, u_N = l_1$$

where $i = 1, \cdots, N-1$, $0 = x_0 < x_1 < \cdots < x_N = 1$, $a_\pm = \frac{1}{2}(a \pm |a|)$.
They used a special mapping which removes the singularity of $u(x)$ when $\varepsilon \to 0$. For this purpose they estimated the derivatives of $u(x)$. For each type (a), (b), (c), they proved that the difference between solutions $u(x)$ of (i), (ii) and $u_i$ of $(i)'$, $(ii)'$ satisfies an error bound of the form $|u_i - u(x_i)| \leq Mh$, $h_i^n = h^n$, $n > 0$ arbitrary. Numerical results are presented for equations of type (a) and (b).

Kadalbajoo and Reddy [119] gave a boundary value method for solving a class of nonlinear, singularly perturbed, two-point boundary value problems with a boundary

layer on the left end of the underlying interval. By constructing a modified problem with a boundary layer correction, the boundary layer is dealt with separately. The method is iterative on the terminal point of the boundary layer. They obtained a lower bound for the terminal point of the perturbation parameter and discussed some theoretical results of the method for linear problems.

Kamowitz [129] studied the applicability of a multigrid approach to the following class of singular perturbation problems

$$-\varepsilon u'' + b(x)u' = f, \; 0 < x < 1, \; u(0) = u_0 \; , \; u(1) = u_1$$

As a main result he showed that if the original system of equations resulting from the discretization of the above problem is of positive type, then the theoretical results for the multigrid algorithm developed in Kamowitz and Parter [128] and in McCormick and Runge [163] apply. In addition to the analysis, detailed numerical studies are presented.

Based on the zeros of Chebyshev polynomials of the first kind, Bamigbola et al. [21] constructed cubic basis functions in one dimension for the solution of two-point boundary value problems. A general formula is derived for the construction of polynomial basis functions of degree $r$, where $1 \leq r < \infty$. A Galerkin finite element method using the constructed basis functions for the cases $r = 1$, 2 and 3 is successfully applied to three different types of problems, including a singular perturbation problem.

Uzelac and Surla [244] considered a collocation cubic spline difference schemes for a singularly perturbed two-point boundary value problem of second order. This family of schemes includes the Il'in finite difference scheme [107], which is uniformly convergent with first order of accuracy. Further they proposed an implicit scheme from this family, which has a better error estimate than Ilin's scheme.

Sklyar [219] constructed a conservative difference scheme for singularly perturbed differential problems. In the construction a suitable decomposition of a symmetric bilinear form (proved in the paper) is applied. The method is presented for the model problem

$$\varepsilon u'' + au' = f \quad , \quad x \in (0,1) \;\; ; \;\; u(0) = \alpha_0 \quad , \quad u(1) = \alpha_1$$

The coefficients of the scheme are obtained by recursion; the number of iterations depends on $\varepsilon$. The order of convergence is proved to be $O(h^2)$ and is independent of $\varepsilon$.

Mattheij and van-Loon [161] considered the $n$-dimensional linear singular perturbation problem

$$\varepsilon \frac{dx}{dt} = A(t,\varepsilon)x + f(t,\varepsilon) \ , \quad t \in [-1,1] \ , \quad 0 < \varepsilon < \varepsilon_0 \ ; \quad B_{-1}^{\varepsilon} x(-1) = B_1^{\varepsilon} x(1) = b_\varepsilon$$

for fixed $\varepsilon_0$. They assumed that this BVP is uniformly well conditioned, in other words, the solution space $S^\varepsilon$ of the homogeneous system is split into two spaces, one of which is nowhere fast decreasing and the other is nowhere fast increasing. Here they concentrated on a second-order ODE. They used the Riccati transform to obtain a Riccati differential equation and found their results through the solution of the Riccati equation.

Sakai and Usmani [209] applied simple exponential splines to solve singular perturbation problems of the form

$$\varepsilon y''(x) + b(x)y'(x) - d(x)y(x) = f(x), \ 0 \leq x \leq 1, \ y(0) = \alpha \ , \ y(1) = \beta$$

for smooth data functions $b$, $d$ and $f$ subject to the conditions $d(x) \geq 0$, $b(x) \geq B > 0$ on [0,1] for a fixed positive $B$. They proved that the limiting case of their collocation method reduces to the collocation method with the usual quadratic spline. An estimate for the approximation error is given along with some numerical experiments.

Lin [148] developed and analysed a class of difference schemes for solving two-point singularly perturbed boundary value problems of the form

$$\varepsilon y'' - b(x,\varepsilon)y = f(x,\varepsilon) \ , \quad y(0) = a_0 \ , \quad y(1) = a_1$$

for small $\varepsilon > 0$, where $b$ and $f$ are assumed to be bounded and at least continuous in $[0,1] \times (0,1]$, and $b(x,\varepsilon) \geq \delta > 0$. The schemes are constructed in a multiple shooting/collocation-type fashion. The functions $b$, $f$ are approximated by sufficiently smooth functions $B_i$, $F_i$ on subintervals $(x_{i-1}, x_i)$ of $[0,1]$. The approximations could be polynomials obtained from the Taylor series for $b$ and $f$ if these functions are smooth, or could be taken from any other approximation space provided they are uniform to some desired order (e.g., $|B_i(x,\varepsilon) - b(x,\varepsilon)| = O(h^p)$, for some $p > 0$). Note that in collocation the solutions are approximated from some polynomial space, rather than the coefficients. Then the sequence of approximate boundary value problems

$$\varepsilon Y_i'' - B_i(x,\varepsilon)Y_i = F_i(x,\varepsilon) \ , \quad Y_i(0) = \beta \ , \quad Y_i(1) = 1 - \beta \ ; \quad \beta = 0,1$$

is solved and the solutions are used to construct a sequence of basis functions similar to the fundamental solutions in shooting theory. The desired solution is written as a linear combination of basis solutions, and continuity conditions through the first order are enforced at the mesh points $x_i$. These conditions define the three-term recurrence relation from which the unknown parameters in the linear combination are obtained, exactly as the linear systems defined by the multiple shooting conditions are solved in ODE-BVP system theory. Finally he showed that these methods converge uniformly as $O(h^p)$ (with stability constant independent of $\varepsilon$) as the stepsize goes to zero. The approach in this paper generalizes the class of variational difference schemes for singularly perturbed BVPs.

Herceg et al. [95] considered singularly perturbed semilinear selfadjoint two-point boundary value problems, with Dirichlet boundary conditions. Using a Bahkalov-type mesh (which is refined in the layers near the endpoints of the interval), they gave a difference scheme for numerically solving such problems. It is shown that the solution of this difference scheme is amenable to Richardson extrapolation, and that one can thereby obtain 6th-order convergence at each node, uniformly in the singular perturbation parameter.

Stynes [232] considered and analysed a three-point difference scheme for the semilinear boundary value problem

$$\varepsilon y'' + a(x)y' - b(x,y) = 0 \ ; \ \ y(0) = A, y(1) = B$$

Turning points, i.e., zeros of $a(x)$, are admitted. The coefficients of the scheme depend nonlinearly on the grid values of $a$. This is an extension of earlier work of Stynes and O'Riordan [231], to nonuniform meshes. As the main result, linear convergence (uniformly in $\varepsilon$, $0 < \varepsilon \leq 1$) of the scheme in a discrete $L_1$-norm is proved with the dependence on the maximal local stepsize ratio made explicit. The corresponding stability proof relies on the fact that the discrete operator is an $M$-function under certain monotonicity assumptions on the coefficients $a$ and $b$. $L_\infty$ results are stated without proof. He also described a simple adaptive mesh generation procedure.

Vulanović [257] discussed a numerical procedure for the singularly perturbed bound-

ary value problem

$$\varepsilon u'' + b(x,u)u' - c(x,u) = 0 \ ; \ u(0) = U_0 \ , \ u(1) = U_1 \ ; \ 0 \le x \le 1$$

Here $\varepsilon$ is a small positive parameter, and the coefficients $b(x,u)$ and $c(x,u)$ are assumed to satisfy appropriate conditions so that there is a boundary layer at $x = 0$. The procedure is to make a nonuniform change of scale of the independent variable, so as to expand the boundary layer, and then to discretize the problem. The numerical solutions obtained give approximations uniform in the small parameter $\varepsilon$. He had used this technique previously for linear problems (see [254] and [255]). This work is the application of the same technique to a quasilinear problem. The method has the advantage of uniform approximation, but may be slower in convergence than a standard Runge-Kutta procedure.

Surla and Herceg [239] analysed the convergence behaviour of the discrete solution for a singular perturbed boundary value problem having exponential properties in boundary layers. They developed a difference scheme by a collocation method, the main tools being splines in tension. These are spline functions piecewise spanned by $\{1,\ x,\ \exp(\rho_j x),\ \exp(-\rho_j x)\}$ in $[x_j, x_{j+1}]$, where $\rho_j$ is a tension parameter. Under the usual regularity assumption for the refinement they obtained uniform convergence results.

Gartland [85] established the strong uniform stability of the continuous singular perturbation problem

$$-\varepsilon u'' + a(x)u' + b(x)u = f(x) \ ; \ u(0) = A \ , \ u(1) = B \ ; \ 0 < x < 1 \ , \ \varepsilon > 0$$

$a \in L^\infty[0,1]$, $b$ and $f \in L^1[0,1]$ and $a(x) \ge \underline{a} > 0$. Finite-difference discretizations of the continuous problem are then studied and some general stability results are verified for these schemes. He concluded in this paper that finite-difference schemes (i) can be convergent without being close to the exact scheme, (ii) can be consistent without being stable, and vice versa, (iii) can be convergent without satisfying certain stability conditions. Furthermore, stability of the finite-difference schemes can be achieved in ways other than using exponential fitting in regions away from layers of the solution, with the exception of the situations where the possible locations of all layers are not known a priori; in these cases the use of exponential fitting is fully justified and desired.

Vulanović [258] considered several difference schemes for solving a singularly perturbed quasilinear two-point boundary value problem on an aribitrary locally almost equidistant mesh. He examined the stability of these schemes, uniformly with respect to the perturbation parameter, in the discrete $L^1$ and $L^\infty$ norms. He also gave some consistency and convergence analyses.

Gasparo and Macconi [86] gave an initial-value method for the second-order singularly perturbed boundary value problems with a boundary layer at one end point. Their method is similar in spirit to that suggested by Kadalbajoo and Reddy [117] but is different in the sense that the method in [117] depends on the knowledge of the reduced solution whereas this method does not. The idea here is to replace the original two-point boundary value problem by two suitable initial value problems which can then be solved efficiently. One drawback of this method is that the integration of these initial value problems goes in opposite direction, and the first problem can be solved only if the solution of the second one is known.

Cash [45] made a comparison between three codes which implement global methods for solving two point boundary value problems. The codes compared are a collocation method, COLSYS; a deferred correction method, PASVA3 and a deferred correction method based on mono-implicit Runge-Kutta formulae, HAGRON. He tested these codes on 12 singular perturbation problems whose solutions have a different type of behaviour(possessing boundary layer/interior layer/corner layer/oscillatory solutions, etc.). The general conclusions which he could draw are the following:

(1) HAGRON is normally faster than either COLSYS or PASVA3 on problems which might be termed mildly stiff to stiff.

(2) COLSYS has the ability to recognize when a singularly perturbed problem has a smooth solution.

(3) COLSYS normally requires fewer grid points than HAGRON and becomes increasingly efficient as the problem becomes stiffer.

(4) The error estimation technique used by PASVA3 is more reliable on these test problems than those used by the other codes. In particular, the COLSYS error estimate can be poor at crude tolerances.

(5) PASVA3 performed well when very high accuracy was required due to its ability to

use high order formulae.

Thus these three codes should be regarded as complementing each other than competing.

Strehmel et al. [228] considered singularly perturbed initial-value problem

$$\varepsilon z'(t) = g(t,y,z) \ , \ z(t_0) = z_0 \ ; \ \varepsilon y'(t) = f(t,y,z) \ , \ y(t_0) = y_0 \ ; \ 0 < \varepsilon << 1$$

where $g : [t_0, t_e] \times R^{n-N} \times R^N \to R^N$ and $f : [t_0, t_e] \times R^{n-N} \times R^N \to R^{n-N}$ are sufficiently differential functions which may also depend smoothly on the small parameter $\varepsilon$. They separated the stiff and nonstiff parts, and used a partitioned linearly implicit Runge-Kutta method. This compound method consists of a customary explicit Runge-Kutta method for the nonstiff and of a linearly implicit Runge-Kutta method(see, Strehmel and Weiner [227]) for the stiff subsystems. Therefore, at each integration step only systems of linear equations of dimension $N$ must be solved. They also considered the limiting case $\varepsilon \to 0$, which gave an index-1 differential-algebraic system. They derived convergence results and gave direct estimates of the local and global error. Some nonlinear test problems are solved.

Based on the Hermitian approximation of the second derivative on special discretization mesh, Herceg [96] studied the numerical solution of a nonlinear two point boundary value problem. A uniform fourth order scheme, which uses a nonuniform mesh to obtain more mesh points in the boundary layer regions, is constructed. He showed that it is not necessary to know the asymptotic expansion of the solution in order to construct special meshes. It is sufficient to know the location of boundary layers and estimates of derivatives of the solution.

In [97] Herceg used the same technique as in [96] for a linear singularly perturbed nonlocal problem

$$\varepsilon^2 u'' + b(x)u = f(x) \ , \qquad 0 \le x \le 1$$

$$u(0) = 0 \ , \ u(1) = \sum_{i=1}^{m} c_i u(s_i) + d$$

$d, c_i \in R$, $s_i \in (0,1)$, $i = 1, 2, \cdots, m$, $0 < \varepsilon << 1$, $b \in C^k[0,1]$, $k \in N$, $b(x) \ge \beta^2 > 0$ for some positive constant $\beta$. The fourth order of uniform convergence as in [96] is proved.

Roos [203] studied the singularly perturbed selfadjoint boundary value problem

$$-\varepsilon^2 u''(x) + p(x)u(x) = f(x) \ , \ u(0) = u(1) = 0$$

where $\varepsilon$ is a small, positive parameter and $p(x)$ satisfies $p(x) \geq p_0 > 0$. Moreover, $p(x)$ and $f(x)$ are continuous functions. The solution of the above problem is approximated by global (that means for all values of $x$), uniformly convergent discretization methods. The basic idea consists in replacing $p(x)$ and $f(x)$ by piecewise polynomials and solving the resulting problems exactly.

Adzic [5] solved the two-point boundary layer problem

$$-\varepsilon^2 y''(x) + F(x)y(x) = G(x) \ , \ \ y(0) = \alpha \ , \ \ y(1) = \beta$$

The solution of this problem is represented in the form

$$y(x) = y_r(x) + u(x) + v(x)$$

where $y_r$ is the solution of the reduced problem and the layer functions $u(x)$ and $v(x)$ are approximated by truncated orthogonal series on the appropriate layer subintervals, determined by the numerical layer length, which depends on the parameter $\varepsilon$ and on the degree of the spectral approximation. The coefficients in the truncated orthogonal series are evaluated using the tau method. An upper bound for the error function is constructed using the principle of inverse monotonicity.

Stojanovic [224] considered a linear, selfadjoint, singularly perturbed, two-point boundary value problem. She generated a difference scheme for this problem by approximating the forcing term with a piecewise cubic polynomial, and approximating the coefficient of the zero-order term with a piecewise constant function. This scheme is shown to be second order accurate, uniformly in the singular perturbation parameter.

Adeniyi [4] presented a recursive formula for a fast and reliable computational error estimate of the tau method for linear ordinary differential equations. The formula is further applied to some nonlinear initial and boundary value problems including a singularly perturbed problem with multiple solutions. The order of accuracy of the tau approximation is estimated through some numerical examples.

Roos and Stynes [204] considered a singularly perturbed fourth-order two-point boundary value problem on $[0, 1]$. Using a variational formulation, they showed how to generate an approximate solution on an arbitrary mesh which is first order accurate, uniformly in the perturbation parameter $\varepsilon$, in the $H^1(0, 1)$ norm. An iterative variant of this method,

closely related to defect correction, yields solutions to any desired order of accuracy in $H^1(0,1)$, uniformly in $\varepsilon$.

The Engquist-Osher(EO) scheme is one of the best known scheme for the numerical solution of quasilinear singular perturbation problems of the following type:

$$-\varepsilon u'' - b(u)u' + c(x,u) = 0 \ , \ \ 0 \leq x \leq 1 \ ; \ \ u(0) = U_0 \ , \ u(1) = U_1$$

There already exists a second-order modification of the EO scheme in the case of equidistant meshes(see, e.g., Lorenz [154]). Vulanović [259] gave nonequidistant generalization of that scheme which he called nonequidistant EOL scheme. Numerical results in [259] show that the nonequidistant generalization of EOL scheme has the pointwise convergence of the first order only, the second order accuracy is presented only at the points outside of the layers. Motivated by this fact, he gave another scheme with better pointwise accuracy. His scheme switches smoothly between the standard central scheme and two midpoint schemes in dependence on the cell Reynolds number. All the schemes are uniformly stable in a discrete $L^1$ norm which is proved by him using the technique of $M$-operators([154], [258]).

Schmitt [212] constructed a symmetric difference scheme for linear, stiff, or singularly perturbed boundary value problems of first order with constant coefficients. His scheme is based on a stability function containing a matrix square root. Its essential feature is the unconditional stability in the absence of purely imaginary eigenvalues of the coefficient matrix. He proved local damping of errors, uniform stability, and uniform second order convergence. He also discussed the computation of the specific matrix square root by a well known stable variant of Newton's method.

In Cash [43] and Cash [44], a deferred correction method is described and analysed that is suitable for the numerical solution of the general first order, nonlinear, TPBVP

$$y' = f(x,y) \ , \ a \leq x \leq b \ , \ g(y(a), y(b)) = 0 \tag{1.5}$$

The motivating idea behind that approach was to combine the desirable properties of the two other widely used approaches to the solution of (1.5): collocation as typified by COLSYS (Ascher et al. [12], [13]) or its variant COLNEW (Bader and Ascher [20]); and iterated deferred correction as typified by the PASVA series of codes (see, e.g.,

Pereyra [193]) based on the approach of Lentini and Pereyra [144]. The unifying theme connecting collocation and deferred correction for first order systems is the use of implicit Runge-Kutta methods. The COLSYS codes are based on an expensive high-order fully implicit Runge-Kutta method defined on Gauss quadrature points. The PASVA codes are based on the trapezium rule, which is a second-order Runge-Kutta method defined on Lobatto points; increased accuracy is obtained through deferred corrections. Intermediate between these two approaches is one that uses a different class of Runge-Kutta method—one that is more expensive than the trapezium rule of PASVA, but less expensive than the high-order method of the COLSYS codes. Increased order with such a method is obtained using deferred correction. With this approach in mind, the new class of mono-implicit Runge-Kutta method was derived (see, Cash [41], [42]). The method developed in [43] and [44] has been implemented in Cash and Wright [46] on both smooth and increasingly stiff problems using their code HAGRON.

Lin [149] considered the following two-point boundary value problem

$$\left.\begin{array}{l} \varepsilon y'' + b(x,y)y' - c(x,y) = 0\ , \quad -1 < x < 1 \\ y(-1) = A\ ,\ y(1) = B \end{array}\right\} \tag{1.6}$$

where $0 < \varepsilon << 1$ and the functions $b(x,y)$ and $c(x,y)$ are assumed to satisfy

$$\left.\begin{array}{l} b(x,y), c(x,y) \in C^2([-1,1]\times R) \\ b(0,y) = 0, b_x(0,y) < 0 \ \text{ for } \ |y| \le r+1 \\ b(x,y) \ne 0 \ \text{ for } \ |y| \le r+1 \ \text{ and } \ x \ne 0 \\ c_y(x,y) \ge c_0 > 0 \ \text{ on } \ [-1,1]\times R \end{array}\right\} \tag{1.7}$$

where $r$ is a finite positive number and $c_0$ is a constant. Earlier he (with Su [148]) modified the initial value technique used by Kadalbajoo and Reddy [117] for (1.6) without turning points and gave an algorithm whose accuracy was good for arbitrary $\varepsilon > 0$. In this paper, he numerically solved the problem (1.6) under the conditions (1.7) using the initial value technique proposed in [117]. He obtained estimates for the solution of (1.6) and its derivatives. He reduced the problem to the solution of two nonlinear Cauchy problems that are approximable by the corresponding difference analouges.

Gonzalez [89] gave an analytic difference to integrate the singularly perturbed scalar problem

$$\varepsilon y'(x) = f(x,y)\ , \quad x \in [a,b]\ ;\ y(a) = y_0\ , \quad 0 < \varepsilon \ll 1$$

The algorithm is based on the exact integration of a locally linearized problem (on a special nonuniform mesh) exhibiting uniform convergence in $\varepsilon$ (for any $x$). The scheme is exact when $f = a + bx + cy$, where $a, b, c$ are constants.

Stojanovic [225] generated an exponential spline difference scheme for a one dimensional singularly perturbed selfadjoint problem along with general boundary conditions of the third kind. She obtained $O(h \min(h, \sqrt{\varepsilon}))$ accuracy if one has Dirichlet's boundary conditions at the right end point, and order one in the remaining cases.

Boglaev and Sirotkin [33] considered and applied two domain decomposition algorithms, A1 and A2, to singularly perturbed boundary value problems. A1 is the standard Schwarz alternating procedure, while A2 is a parallel algorithm obtained from A1 by introducing an interfacial problem. They proved some convergence results for 1-D and 2-D semilinear problems and presented some numerical results, including a 2-D example with an interior layer.

A stabilized treatment of the following class of singular perturbation problems with spectral methods is presented in Eisen [70]:

$$-\varepsilon u'' + u' = f \ \text{ in } \ (-1, 1); u(-1) = u(1) = 0, 0 < \varepsilon \leq 1$$

The stabilization is achieved by introducing one additional collocation point. The resulting overdetermined system leads to the normal equations which yield a stabilized system. They also presented some numerical results in one and two dimensions which show the improved condition number of the stabilized spectral system.

Sun and Wu [233] proposed a scheme for the numerical solution of the singular boundary value problem

$$\varepsilon u'' + b(x)u' - c(x)u = f(x) \ \ , \ \ u(0) = \alpha \ \ , \ \ u(1) = \beta$$

Their scheme is obtained by coupling the central difference scheme and the Abrahamsson-Keller-Kreiss box scheme on a special nonuniform mesh. They proved that this scheme is uniformly second-order convergent.

Heinrichs [98] gave a method of stabilization for finite difference or spectral approximation of singular perturbation problems. The original domain $(-1, 1)$ is split into $(-1, 1 - c\varepsilon)$ and $(1 - c\varepsilon, 1)$, where $c$ is a positive constant chosen in a suitable way. Using the artificial viscosity and perturbing the right-hand side by adding first and higher

derivatives, he obtained stable discrete operators. The approximation errors, away from the boundary layer, have the same order as the corresponding discretization errors. He also proposed some modifications of the method for applications to multi-grid techniques. He gave some numerical results for the one- and two-dimensional cases.

Petrovic [194] considered a semilinear singularly perturbed second-order, selfadjoint two-point boundary value problem. He constructed a finite difference scheme on an exponentially graded mesh of Bakhvalov type, and showed that this scheme has a unique solution, which is a point of attraction for the Newton's method. Writing $h$ for the mesh diameter and $\varepsilon$ for the singular perturbation parameter, he proved that, provided sufficiently many mesh points are used, the nodal errors in the computed solution are bounded by $C(h^4 + \varepsilon^2)$, where the constant $C$ is independent of $\varepsilon$ and of the mesh. Corroborating numerical results are presented for a linear problem.

Stojanovic [226] gave a piecewise polynomial finite-element method which uses a piecewise linear approximation for driving terms in solving a nonselfadjoint perturbation problem. She derived a three-point difference discretization scheme and proved its second-order uniform convergence.

Russo and Peskin [207] described a computational procedure which is designed to analyze automatically the behavior of certain general classes of nonlinear singular perturbation problems by applying the combined results of a body of theory that proves the existence of solutions for these problems. They created a computer program that implements the computational procedure. The core mathematical knowledge contained in their program is composed of rules that embody the results of mathematical theorems from nonlinear singular perturbation theory. The principal method of proof yields an estimate of a solution by constructing sharp bounding functions that define a region in which the solution exists uniquely. The ability to construct such a program depends critically on the successful coupling of a nondeterministic programming technique called path-finding with the capabilities of a computer algebra system.

Using fourth order cubic spline, Kadalbajoo and Bawa [121] solved some linear singularly perturbed initial value problems. Their method consists of dividing the inner and outer region and solving the inner region problem by a fourth order cubic spline after rescaling the inner region. The solution of the reduced problem provides the solution of

the outer region problem. The point of division of inner and outer region is arbitrary. The solution obtained at this point is discontinuous. So to overcome this drawback they gave a cutting point technique also.

A survey on control problems where the state equation is linear with respect to the phase variable and the derivative is multiplied by a constant matrix can be found in Kurina [141]. In this paper the connection with various singularly perturbed control problems is discussed.

Kadalbajoo and Bawa [122] presented a variable-mesh method based on cubic spline approximation for nonlinear singularly perturbed boundary-value problems of the form

$$\varepsilon y'' = f(x,y) \ , \ \ y(a) = \alpha \ , \ \ y(b) = \beta$$

They gave convergence analysis and the method is shown to have third-order convergence.

By a generalization of the SQRT scheme introduced by Schmitt [212], stiff systems of linear ODEs are solved numerically by Schmitt and Mei [213]. Their method uses one-step schemes of Runge-Kutta type with the weights being functions of the coefficient matrix of the system. The stability function of the schemes is of algebraic type. A-stability of the method and its performance when applied to singularly perturbed problems are discussed.

A domain decomposition algorithm for a singular perturbation problem in one dimension has been analyzed and illustrated by Boglaev [34]. They first analyzed the decomposition to two subintervals and then the number of subdomains is increased. In all cases the domains are overlapping. The idea is to have the boundary layers localized; then regular grids in each subdomain can be used.

Reusken [200] considered a multigrid method applied to a class of singularly perturbed two-point boundary value problems. In this method he used a matrix-dependent prolongation and restriction. For a class of two-grid methods he proved uniform convergence for all $h$ (mesh size parameter) and $\varepsilon$ (perturbation parameter).

Attractive turning point singularly perturbed problems (with internal layers) are discussed in Clavero and Lisbona [52]. They analyzed the uniform convergence (in $L^1$ norm) of three finite difference schemes on irregular meshes for the solution of such problems. They discussed both linear and semilinear ordinary differential equations.

Vulanović [261] considered a two-point boundary value problem

$$-\varepsilon u''(x) - b(u)u'(x) + c(x,u) = 0 \quad \text{on } (0,1),$$

where $u(0)$ and $u(1)$ are given, $0 < \varepsilon \ll 1$, the functions $b$ and $c$ are smooth, $c_u(x,u) \geq c_* > 0$ on $[0,1] \times R$, and $b(u) \geq b_* > 0$ on $[u_*, u^*]$, where $u^*$ and $u_*$ are upper and lower solutions respectively of the given problem. He surveyed several papers that discuss numerical methods for this problem. A central difference scheme on a special graded mesh is then shown to be first-order convergent, uniformly in $\varepsilon$, in the discrete maximum norm.

Kramer [138] proposed a divide and conquer method. It employs a coarse grid discretization, as multiple shooting does, but solves the local problems in a 'BVP-way', i.e., by a global method. A sophisticated error control is developed to combine local and global convergence of the Newton updating. An implementation based on using the collocation code COLNEW is briefly discussed and a number of examples are given to illustrate the success of the method, in particular for singular perturbation problems.

Imanaliev and Pankov [108] considered the singularly perturbed problem

$$\varepsilon y'(t) = f(t,y) \ , \quad t \in [0,\infty) \ , \quad y(0) = y_0 \in R$$

where $\varepsilon$ is a small positive parameter. They studied the rise of an interior boundary layer for this problem. They introduced the notion of the moving interior boundary layer and found conditions for the appearance of such a layer. It is shown that in the vector case it is possible for more complicated phenomena to arise, such as a combination of withdrawing and rotating boundary layers.

Vulanović et al. [262] gave some numerical methods for the following three singular perturbation problems:

$$-\varepsilon u'' + u^r = 0 \tag{1.8}$$

$$-\varepsilon u'' + \varepsilon\theta(1+\theta u)^{-1}u'^2 + (1+\theta u)u^r = 0 \tag{1.9}$$

$$-\varepsilon u'' + \varepsilon\theta(1+\theta u)^{-1}u'^2 - 2\varepsilon x^{-1}u' + (1+\theta u)u^r = 0 \tag{1.10}$$

subject to the same conditions

$$-u'(0) = 0 \ , \ \ u(1) + \sigma u'(1) = 1 \ , \ \ x \in (0,1) \tag{1.11}$$

where $r$ and $\sigma$ are nonnegative constants, $0 < \varepsilon \ll 1$. These problems model some catalytic reactions. They considered only the more difficult case $\sigma = 0$. When $\sigma = 0$, $u_\varepsilon$ has a layer at $x = 1$. First, they introduced a general linearization procedure which produces a monotonically decreasing sequence of upper solutions. Its convergence towards the exact solution is proved for the simplest problem (1.8),(1.11). The convergence is uniform in $\varepsilon$. Each of the linear problems can be solved by using finite differences on special nonuniform meshes which are dense in the layer $x = 1$. The proof of the uniform accuracy of the numerical results is not given.

Mazzia and Trigiante [162] discussed a three-point difference method for the solution of the singularly perturbed two-point boundary value problem

$$\varepsilon y''(t) + s(t)y'(t) + c(t)u(t) = f(t) \ , \ \ y(t_{\min}) = a \ , \ \ y(t_{\max}) = b$$

which is of second order for equidistant meshes. Four different conditions are given implying stability of the scheme, the simplest one being

$$c(\zeta) \le s(t_{i+1})s(t_i)/(4\varepsilon) + s'(\eta)/2 \ , \ \ c(t_i)c(t_{i+1}) \ge 0 \ ; \ \ \zeta, \eta \in [t_i, t_{i+1}]$$

where $t_i, t_{i+1}$ are mesh points. A mesh selection strategy is described which uses these stability conditions and representations of the local truncation error. The construction of the mesh starts with the roots of $s$ and the boundary points, where layers may be present. In numerical experiments on five test problems with interior layers, this algorithm is compared with the code COLNEW. The superior performance of the new algorithm for small values of $\varepsilon$ is mainly attributed to the mesh selection scheme.

Schneider [214] mentioned that many numerical methods used to solve ordinary differential equations or differential algebraic equations can be written as general linear methods. The $B$-convergence results for general linear methods are for algebraically stable methods, and therefore useless for nearly $A$-stable methods. Schneider showed convergence for singular perturbation problems for the class of general linear methods without assuming $A$-stability.

It is well known that many important classes of Runge-Kutta methods suffer an order reduction phenomenon when applied to certain classes of stiff problems. In particular, the $s$-stage Gauss methods with stage order $s$ and order of consistency $2s$ behave like methods of order $s$ when applied to the class of singularly perturbed problems. Burrage and Chan [37] showed that the process of smoothing can ameliorate this effect, when dealing with initial value problems. They did this by first studying the effect of smoothing on the standard Prothero-Robinson problem and then by extending the analysis to the general class of singularly perturbed problems.

Selvakumar [217] considered a linear singularly perturbed two-point boundary value problems with a boundary layer on the left end of the interval. The original problem is divided into inner and outer region problems. The zeroth-order asymptotic expansion is used to obtain the terminal boundary condition. Then, a new inner region problem is created and solved as a two-point boundary value problem (TPBVP). In turn, the outer region problem is also solved as a TPBVP. Both these problems are efficiently solved by employing uniform and optimal exponentially fitted finite difference schemes.

Wang [266] solved a nonlinear singular perturbation problem numerically on nonequidistant meshes which are dense in the boundary layers. The method is based on the numerical solution of integral equations (see, Delves and Mohamed [58]). He proved the fourth-order uniform accuracy of the scheme.

In relation to boundary value problems, Roos [205] proposed the following approach: first solve an auxiliary problem with piecewise or nearly piecewise constant coefficients; and then, improve the approximation iteratively using the defect correction idea and piecewise polynomial approximations of higher order. An important advantage of this approach lies in the fact that it is possible to start from a classical or weak formulation of a boundary value problem. Therefore, the approach is useful for singular perturbations related to ordinary as well as partial differential equations.

Vulanović [264] considered a singularly perturbed boundary value problem

$$-\varepsilon^2 u''(x) + c(x,u) = 0 \ , \quad x \in [0,1] \ , \quad u(0) = \gamma_0 \ , \quad u(1) = \gamma_1$$

without assuming a uniform bound $\partial c/\partial u \geq c_* > 0$. He gave some results for a special subclass with $c(x,u) = a(x,u)u^r$, $r \geq 2$. Several assumptions assure a positive solution

for which explicit lower and upper bounds are presented. The numerical scheme uses finite differences and an analytically given mesh that is dense in boundary layers. With the help of a lower estimate $\partial c/\partial u \geq (\text{const})\,\varepsilon^2 > 0$; the M-matrix properties are used to prove the error estimate.

Miller et al. [167] gave a survey of their own results concerning difference methods for the numerical solution of singularly perturbed boundary value problems. The model problem

$$\varepsilon u_{xx} + u_x = 0 \ , \quad u(0) = 0 \ , \quad u(1) = 1 \ ; \quad \varepsilon > 0$$

and some simple (not exponentially fitted) corresponding difference schemes are considered for illustration. A class of piecewise-uniform meshes is proposed in which a finer mesh size $h_1$ is used for the leading $N/2$ grid points, and a coarser mesh size $h_2$ for the remaining $N/2$ grid points. They showed that the difference schemes considered are $\varepsilon$-uniform only if the transition point between the two mesh sizes depends on $N$ and $\varepsilon$ in a specific way.

In many practical situations the application of multigrid methods to finite element approximations calls for the deployment of grids with high aspect ratios. In these situations the elements become degenerate in the sense that the smallest angle tends to zero as the grids are refined and, as a consequence, the convergence rate of conventional multigrid methods deteriorates. Keeping these situations in mind Zhang [271] devised and analysed nonnested multigrid methods for model problems (including singular perturbations, interface problems and convection-diffusion problems) on triangular grids, each element having one horizontal and one vertical edge. He proved that the convergence rate of these methods is not impaired by high aspect ratios.

Grekov and Krasnikov [91] examined a linear singularly perturbed reaction-diffusion problem in one dimension. Assuming that its coefficients are piecewise smooth, they considered any mesh whose nodes include the points of discontinuity of these coefficients. The solution $u$ is expressed as a series, each term of which can be computed by numerically solving a singularly perturbed reaction-diffusion problem with piecewise constant coefficients. They proved that by truncating this series, $u$ can be approximated in the $L^\infty$ norm, uniformly in the singular perturbation parameter, up to $O(h^m)$, where $h$ is the mesh diameter and $m$ is an arbitrary positive integer.

Hu et al. [102] developed a discritization method for one dimensional singular perturbation problems based on Petrov-Galerkin finite element, or an equivalent finite volume, scheme. The model one dimensional problem which they considered was

$$\left.\begin{array}{l} -\varepsilon u'' + \beta u' + \sigma u = f \ \text{ in } \ (a,b) \\ u(a) = u_a \ , \ u(b) = u_b \end{array}\right\} \tag{1.12}$$

This problem has its origin in the physical conservation law

$$q' + (\sigma - \beta')u = f \tag{1.13}$$

and Fick's diffusion law

$$q = -\varepsilon u' + \beta u \tag{1.14}$$

where, $q$ is the flux, $\varepsilon$ the diffusivity, $\beta$ the velocity, and $\sigma$ the absorbing coefficient (or reactivity). Till this work the only reference to a second-order scheme for the conservation form (1.13)-(1.14) of (1.12) was O'Riordan and Stynes [187] but that scheme was not discretely conservative. However, in the above work of Hu et al., the developed numerical scheme is not only $O(h^2)$ accurate uniformly in $\varepsilon > 0$, but also satisfies certain discrete versions of both the conservative law and maximum principle.

Ishiwata and Muroya [109] considered the linear system $Ax = b$ derived from singular perturbation problems. To solve such non-symmetric linear problems, they proposed a generalized SOR method, which they named as 'improved SOR method with orderings'. They used three ideas, that is, orderings, variable relaxation parameters and a not necessarily strictly upper triangular splitting matrix $U$ in $A = D - L - U$.

Cash et al. [47] developed an automatic continuation algorithm for the solution of linear singular perturbation problems. They incorporated this into two codes which implement global methods for solving two-point boundary value problems. Their algorithm is based upon error estimates formulated in the collocation code COLSYS and the deferred correction code HAGRON. The benefits of using continuation are clearly demonstrated for both codes for a large class of problems.

Kalachev and Mattheij [127] discussed the questions of well-conditioning and dichotomy (see, de Hoog [56]) for the boundary value problem

$$\varepsilon y'' + a(x)y' + b(x)y = f(x) \ , \ x \in [x_0, x_1] \ , \ y(x_0) = \alpha \ , \ y(x_1) = \beta$$

with $\varepsilon$ positive and small. For this analysis, they relate the original problem to another one for an "optimally scaled" system of first-order equations. The case $a(x) \equiv 0$ is investigated using WKB asymptotic formulas. It is shown that if $b(x) < 0$ then the problem is dichotomic and well-conditioned, but if $b(x) > 0$ (or $b(x)$ changes sign) the answer is the opposite. They also obtained a criterion for dichotomy for the case $a(x) \neq 0$.

A survey of finite-difference methods for singularly perturbed boundary value problems with quadratic nonlinearity can be found in Vulanović [265].

Cash [48] surveyed the use of implicit Runge-Kutta methods for the numerical solution of singular perturbation problems. He showed that certain classes of implicit Runge-Kutta formulae are both stable and accurate for singularly perturbed problems. He also described the implementation of these formulae (based on the use of deferred correction).

Tang and Trummer [243] considered a pseudospectral methods (PS) for the numerical solution of

$$\varepsilon u'' + p(x)u'(x) + q(x)u(x) = f(x) \ , \ \ -1 < x < 1 \ ; \ \ u(-1) = \alpha \ , \ \ u(1) = \beta$$

where $\varepsilon > 0$ is small. Accuracy requires that boundary layers of width $O(\varepsilon)$ be resolved, which means $N$, where the gridpoints for PS are $x_j = \cos(\pi j/N)$, must satisfy $N = O(\varepsilon^{-1/2})$. If $\varepsilon \ll 1$, the method becomes impractical. They made the change of variables $x = g_m(y)$, where $g_0(y) = y$ and $g_m(y) = \sin(\frac{\pi}{2} g_{m-1}(y))$ and applied PS to the transformed problem. The transformation moves some collocation points much closer to the endpoints and in the layers. They gave a heuristic argument as to why numerical computations are less sensitive to round-off than a condition number of $O(N^4)$ might suggest.

Liu and Shen [151] solved a time-independent convection-diffusion equations, where the diffusion term is multiplied by a small parameter. They gave a modified Legendre-Galerkin method. It consists in first applying a suitable transformation to the original equation, and then in using an appropriately chosen space of trial functions. The construction of this trial space is given in detail in one and two spatial dimensions. Along with the theoretical results on the rate of convergence of the proposed scheme, some numerical examples are also discussed.

Garbey [82] discussed a Schwarz alternating procedure for singular perturbation prob-

lems. He showed that the procedure offers a good algorithm for the numerical solution of these problems, provided the domain decomposition is properly designed to resolve the boundary and transition layers. He gave sharp estimates for the optimal position of the domain boundaries and the convergence rates of the algorithm for various linear second-order singular perturbation problems. Some implementation results for a turning-point problem and a combustion problem were also reported on.

Axelsson and Nikolova [19] gave a difference method for singularly perturbed convection-diffusion problems with discretization error estimates of high order (order $p$), which hold uniformly in the singular perturbation parameter $\varepsilon$. The method is based on the use of a defect-correction technique and special adaptively graded and patched meshes, with mesh sizes varying between $h$ and $\varepsilon^{3/2}h$ when $p = 2$, where $h$ is the mesh size used in the part of the domain where the solution is smooth, and $\varepsilon^{3/2}h$ is the final mesh size in the boundary layer. Similar constructions hold for interior layers. The correction operator is a monotone operator, which makes it possible to estimate the error of optimal order in the maximum norm. The total number of mesh points used in a $d$-dimensional problem is $O(\varepsilon^{-s})h^{-d} + O(h^{-d})$, where $s$ is $1/p$ or $1/2p$, in the case of boundary or interior layer, respectively.

Lee and Greengard [143] described an adaptive algorithm for the solution of singularly perturbed two point boundary value problems. Working with an integral equation reformulation of the original differential equation, they introduced a method for error analysis which can be used for mesh refinement even when the solution computed on the current mesh is underresolved. Based on this method, they constructed a black-box code for stiff problems which automatically generates an adaptive mesh resolving all features of the solution. The solver is direct and of arbitrary high-order accuracy but requires an amount of time proportional to the number of grid points.

Adzic [6] considerd a selfadjoint singularly perturbed problem described by a second-order differential equation. The solution inside the layer is approximated by Newton's iteration represented in the form of a truncated orthgonal series with respect to the Chebyshev basis. For that purpose, domain decomposition is performed according to a suitable resemblance function. The coefficients of the spectral approximation are determined by a collocation method at Gauss-Lobatto nodes. The error function is estimated

according to the principle of inverse monotonicity, using the asymptotic behavior of the exact solution.

Natesan and Ramanujam [174] solved some singularly-perturbed turning-point probems exhibiting twin boundary layers using the initial-value technique that was developed for solving singularly-perturbed nonturning-point problems by Gasparo and Macconi [87]. They obtained the required approximate solution by combining solutions of the reduced problem, an initial-value problem, and a terminal-value problem. They also discussed the implementation of the method on parallel architectures.

Kadalbajoo and Rao [123] gave a parallel discrete invariant embedding algorithm for solving singularly perturbed boundary value problems. They presented this algorithm for both unlimited and limited processors using hypercube architecture.

Using the adaptive finite difference method of Huang and Sloan [103], Mulholland et al. [172] described a method of coordinate transformation that enables spectral methods to be applied efficiently to differential problems with steep solutions. The adaptive finite difference solution permits the construction of a smooth coordinate transformation that relates the computational space to the physical space. The map between the spaces is based on Chebyshev polynomial interpolation. They applied the standard pseudospectral (PS) method to the transformed differential problem and obtained nonoscillatory numerical solutions for steady problems in one and two space dimensions.

Farrell et al. [79] considered come discrete approximations for certain classes of singularly perturbed boundary value problems. They showed that for representative classes of singular perturbation problems there exist no fitted operator schemes on uniform grids, whose solutions converge, in the uniform grid norm, $\varepsilon$-uniformly to the solutions of the boundary value problems. To illustrate this, they presented and discussed results of numerical experiments for a semilinear problem.

Kopteva [137] considered the singularly perturbed boundary value problem in the conservation form $Lu \equiv -\varepsilon(p(x)u')' - (r(x)u)' + q(x)u = f(x), 0 < x < 1, u(0) = g_0, u(1) = g_1, p(x) \geq p_0 = \text{const} > 0,\ r(x) \geq r_0 = \text{const} > 0,\ q(x) \geq 0$, where $\varepsilon \in (0,1]$ is a small parameter. He investigated the difference scheme with the central approximation of the convection term $r(x)u$ and studied convergence properties of this difference scheme on two layer-adopted meshes. On the logarithmically graded mesh

(Bakhvalov-type mesh), he proved that the difference scheme converges uniformly in the perturbation parameter $\varepsilon$ with a convergence rate $O(N^{-2})$. This estimate holds true on a modification of Bakhvalov-type mesh, where the requirement on the smoothness of the mesh-generating function is omitted. He also showed that the difference scheme on the piecewise equidistant mesh (Shishkin-type mesh) converges uniformly with a convergence rate $O(N^{-2}\ln^2 N)$.

## 1.4 Summary of the Thesis

We considered mainly the one dimensional singularly perturbed second order two point boundary value problems. Methods are devised for linear problems, nonlinear problems, turning point problems having boundary layers and the turning point problems having interior layers. The developed methods have been analysed for convergence and it has been found that all the methods are second order accurate. Numerical experiments are also given.

There are three principal approaches to solve numerically these type of problems, namely, the Finite Difference Methods, the Finite-Element Methods and the Spline Approximation Methods. It can be observed from Section 1.3 that the first two have been used by numerous researchers. We used the third approach, namely, the Spline Approximation Methods, to solve the problems of the above type.

The present thesis deals with the numerical methods for solving singular perturbation problems (SPPs). This thesis is divided into seven chapters. We consider mainly the one dimensional singularly perturbed second order two point boundary value problems. Methods are devised for linear problems, nonlinear problems, turning point problems having boundary layers and the turning point problems having interior layers. The methods developed have been analysed for convergence and it has been found that all these methods are second order accurate. Several numerical experiments are given at the end of each chapter.

The first chapter reviews in detail the various numerical methods to solve the one-dimensional SPPs. This review contains a surprisingly large amount of material and indeed can serve as an introduction to some of the ideas and methods of singular pertur-

bation theory. Starting from Prandtl's work a large amount of work has been done in the area of singular perturbations. This chapter limits its coverage to some standard singular perturbation models considered by various workers, the numerical methods developed by numerous researchers to solve SPPs and to the summarry of this thesis.

Consider the following class of SPPs:

$$\left.\begin{array}{l} Ly \equiv \varepsilon y'' + a(x)y' + b(x)y = f(x) \\ y(0) = \alpha_0 \ , \ y(1) = \alpha_1 \ ; \ \alpha_0, \alpha_1 \in R \end{array}\right\} \tag{1.15}$$

where, $a(x)$, $b(x)$, $f(x)$ are sufficiently smooth with $a(x) > c > 0$, $b(x) > 0$, $c$ is some constant and $\varepsilon$ is a small positive parameter.

It is well known that the solution of the problem (1.15) when $a(x) \equiv 0$ has oscillatory behaviour. To overcome these oscillations, in chapter 2, we develop a method using spline in compression. In this chapter we consider three types of problems. First we analyse the problems in which the second derivative term and the function term in (1.15) are present while the term containing the first derivative is absent. The problems having the second and first derivative terms but lacking the function term are considered in the second case. Finally the third case deals with the most general problem. By making use of the continuity of the first order derivative of the spline function, the resulting spline difference scheme gives a tridiagonal system which can be solved efficiently by the well known algorithms.

For the problems of the type

$$\left.\begin{array}{l} Ly \equiv \ - \varepsilon y'' + a(x)y' + b(x)y = f(x) \quad \text{on} \quad (0,1) \\ y(0) = \alpha_0, \ y(1) = \alpha_1, \ \alpha_0, \alpha_1 \in R \end{array}\right\} \tag{1.16}$$

where, $x \in (0,1)$, $\varepsilon$ is a small positive parameter and $a(x)$, $b(x)$ and $f(x)$ are bounded continuous functions, we give another method which is based on spline in tension. It is given in chapter 3.

There are two possibilities to obtain small truncation error inside the boundary layer(s). The first is to choose a fine mesh there whereas the second one is to choose a difference formula reflecting the behaviour of the solution(s) inside the boundary layer(s). Since the spline difference scheme has the same order of precision and the same matrix structure on the uniform and on the non-uniform grid for a fixed $\varepsilon$, we use this property for singularly perturbed problems. This enables us in modifying the distribution of mesh

points vis-a-vis to the properties of the exact solution. Keeping these two ideas in mind, in chapter 4, we give two other methods: one is Variable Mesh Spline in Compression referred to as VMSC and the other is Exponentially Fitted Spline in Compression referred to as EFSC to solve the problems of the type (1.15) when $b(x) \equiv 0$. It has also been explained in the same chapter that why the methods VMSC and EFSC have not been used for the case when both $a(x)$ and $b(x)$ in (1.15) are non-zero.

In chapter 5, we consider the following class self-adjoint singularly perturbed two point boundary value problems

$$\left.\begin{array}{c} Ly \equiv -\varepsilon\,(a(x)y')' + b(x)y = f(x) \quad \text{on} \quad (0,1) \\ y(0) = \eta_0, \quad y(1) = \eta_1 \end{array}\right\} \tag{1.17}$$

where, $\eta_0$, $\eta_1$ are given constants and $\varepsilon$ is a small positive parameter. Further, the coefficients $f(x)$, $a(x)$ and $b(x)$ are smooth functions and satisfy

$$a(x) \geq a > 0, \; a'(x) \geq 0, \; b(x) \geq b > 0$$

The original problem (i.e. problem (1.17)) is reduced to the normal form. We then apply spline in tension to the normal form. To obtain the improved results over this method we give two other methods: one is Variable Mesh Cubic Spline method referred to as VMCS and the other is Exponentially Fitted Cubic Spline method referred to as EFCS. Both these methods are presented in this chapter along with the reason for not using the Spline in Tension with variable mesh/exponential fitting for such problems.

The methods developed in chapters 2 and 3 for linear problems are extended for nonlinear problems in chapter 6. Using the well known quasilinearization method of Bellman and Kalaba (1965), the original nonlinear differential equation has been linearized as a sequence of linear differential equations. Each of these linear equations is then solved by the schemes derived for linear case using spline in compression and spline in tension. (We also use variable mesh cubic spline for some problems). In limit, the solution of these linearized problems converges to the solution of the original nonlinear problem.

Finally, in chapter 7, we consider the following class of singularly perturbed turning point problems

$$\left.\begin{array}{c} Ly \equiv \varepsilon y'' + a(x)y' - b(x)y = f(x) \quad \text{on} \quad [p_1, p_2] \\ y(p_1) = \eta_1, \quad y(p_2) = \eta_2 \end{array}\right\} \tag{1.18}$$

where, $a(x)$ is assumed to be in $C^2[p_1, p_2]$, $b(x)$ and $f(x)$ are required to be in $C^1[p_1, p_2]$, $\eta_1$, $\eta_2$ are given constants, $p_1 \leq 0$, $p_2 > 0$ (usually $p_1 = -1$ and $p_2 = 1$), $0 < \varepsilon << 1$. Moreover

$$a(0) = 0 \quad , \quad a'(0) < 0 \tag{1.19}$$

In order that the solution of (1.18) satisfies a maximum principle, we require that

$$b(x) \geq 0 \quad , \quad b(0) > 0 \tag{1.20}$$

Also $b(x)$ is required to be bounded below by some positive constant $b$, i. e.,

$$b(x) \geq b > 0 \tag{1.21}$$

to exclude the so-called resonance cases. We also impose the following restriction which ensures that there are no other turning points in the interval $[p_1, p_2]$:-

$$|a'(x)| \geq \left|\frac{a'(0)}{2}\right| \quad , \quad x \in [p_1, p_2] \tag{1.22}$$

(If there is no first derivative term but if $b(x)$ changes sign then also turning point occurs, conventionally termed as classical turning point).
Under these conditions (1.19 - 1.22), the turning point problem (1.18) has a unique solution having two boundary layers at $x = p_1$ and $x = p_2$.

Using the usual first order Taylor series approximations for the first order derivative of the approximate solution which we seek using Ahlberg's cubic spline, the continuity condition give us a tridiagonal system. The variable mesh strategies for three different cases, viz., when there are two boundary layers (one at both ends), when there is one boundary layer at the left end and when there is one boundary layer at the right end are given in this chapter.

In the above problem instead of (1.19) if we have the following condition

$$a(0) = 0 \quad , \quad a'(0) > 0 \tag{1.23}$$

then the solution of the problem (1.18) will possess interior layer. We give the treatment of such problems in the same chapter.

## 1.5 Further Developements

It is well known that if any discretization technique is applied to a parameter dependent problem, then the behaviour of the discretization depends on the parameter. For singularly perturbed problems, conventional techniques often lead to discretizations that are worthless if the perturbation parameter is close to some critical value. Also discretization of the problems leads to a linear or nonlinear system of equations with a large number of unknowns. Iterative methods were commonly used to solve these systems. It is important to realize that iterative solvers should be robust with respect to the singular perturbation parameter. The care must be taken while analyzing the dependence on this parameter of those constants that arise in consistency, stability and error estimates.

Numerical grid generation has become an extremely important device for use in efficient computational solution of nonlinear differential equations (both ODE and PDE). When a solution involves near-singular behaviour such as a boundary layer, a shock or a transition or interior layer, it is desirable to have a computational grid that is adapted to these local features. Construction of adaptive grid generators is now an area of intense activity, an activity that finds much of its motivation in computational fluid dynamics. The common theme in much of the work on adaptive grid methods is the idea of equidistribution, which seeks to distribute some function uniformly over the domain of the problem. The function is usually some measure of the computational error. Fundamental to the success of reliable adaptive algorithms is the availability of sharp error estimates. The use of sharp estimates ensures that the adaptive algorithm is efficient in the sense that it does not produce a grid that is over-refined for a specified error tolerance.

In some of the papers in the above review it has been observed that the boundary layers of many singular pertutbation problems are exponential in nature. A number of so called exponentially fitted methods have been proposed by several researchers. These methods aim to produce accurate solutions on relatively coarse grids. Fitted mesh methods are useful for many more classes of problems than those for which theoretical error estimates have been established. The readers are encouraged therefore to use their intuitions to attempt to design an appropriate fitted mesh for their problems, whether it be linear or nonlinear. However, they can be computationally expensive to implement,

especially for nonlinear problems, and are not easily extendible to multidimensions. The second approach is to use a standard discretization on a suitably chosen non-uniform grid. However, the successful application of any non-uniform grid method requires significant amount of a priori information about the presence, location, height and width of the layers. Therefore to obviate the need for unrealistic amounts of a priori information about the solution, adaptive algorithms are to be developed which can detect automatically the presence and thickness of boundary layers.

The computations reported in this thesis were done on Silicon Graphics Origin 200 (dual processor) Operating System (in Fortran 77 in double precesion with 16 significant figures) at IIT Kanpur.

# Chapter 2

## SPLINE IN COMPRESSION FOR SINGULAR PERTURBATION PROBLEMS

## 2.1 Introduction

The numerical techniques for solving singularly perturbed two point boundary value problems can be classified in three broad categories, namely, the Finite Difference Methods, the Finite-Element Methods and the Spline Approximation Methods. It can be observed from the review in Section 1.3 of first chapter of this thesis that the first two have been used by numerous researchers. We used the third approach, namely, the Spline Approximation Methods, to solve such problems.

Consider the singularly perturbed problem

$$\left.\begin{array}{l} Ly \equiv \varepsilon y'' + a(x)y' + b(x)y = f(x) \\ y(0) = \alpha_0 \ , \ y(1) = \alpha_1 \ ; \ \alpha_0, \alpha_1 \in R \end{array}\right\} \tag{2.1}$$

where, $a(x)$, $b(x)$, $f(x)$ are sufficiently smooth with $a(x) > c > 0$, $b(x) > 0$, $c$ is some constant and $\varepsilon$ is a small positive parameter.

This class of problems arise in various fields of science and engineering, for instance, fluid mechanics, quantum mechanics, optimal control, chemical-reactor theory, aerodynamics, reaction-diffusion process, geophysics etc.

There are a wide variety of asymptotic expansion methods available for solving the problems of the above type. But there can be difficulties in applying these asymptotic expansion methods, such as finding the appropriate asymptotic expansions in the inner and outer regions, which are not routine exercises but require skill, insight and experimentations. In view of the wealth of the literature available on singular perturbation problems and in view of the specialised skills and experience that experts in the field

deem necessary, one can raise the question whether there may be other ways to attack these problems, ways that are easy to use and ready for computer implementation, ways that are more accessible to the practicing engineers or applied mathematicians. The spline technique is one such tool to reach these goals in an optimum way.

This type of problems have been solved earlier by several workers. Notably, Chin and Krasny [51], used $\gamma$ -elliptic splines, Flaherty and Mathon [80], used tension splines and Sakai and Usmani [209], used exponential B-splines to solve such problems. However, in none of them a uniform convergence was achieved.

In this chapter we have presented a new approach based on spline in compression. By making use of the continuity of the first order derivative of the spline function, the resulting spline difference scheme gives a tridiagonal system which can be solved efficiently by the well known algorithms. We consider three types of problems. First we analyse the problems in which the second derivative term and the function term are present while the term containing the first derivative is absent. The problems having the second and first derivative terms but lacking the function term are considered in the second case. Finally the third case deals with the most general problem.

In Section 2.2 we give a brief description of the method. The derivation of the difference schemes for all the three cases has been given in Section 2.3. In Section 2.4 we have shown the second order accuracy of the method. In Section 2.5 we have solved five numerical examples to demonstrate the applicability of the method. The discussion on our results together with some comparisons with the earlier results obtained by others is given in Section 2.6.

## 2.2 Description of the Method

For $x \in [x_{j-1}, x_j]$, we define $\widetilde{a}(x) = (a_{j-1} + a_j)/2$ and anologously $\widetilde{b}(x)$ and $\widetilde{f}(x)$ too. Consider first the equation (2.1), with $a(x) \equiv 0$.

The approximate solution of this problem, is sought in the form of the function S(x), which on each interval $[x_{j-1}, x_j]$, (denoted by $S_j(x)$) satisfies the following relations :-

(i) the differential equation

$$\varepsilon S_j''(x) + \tilde{b}(x)S_j(x) = \tilde{f}(x) \tag{2.2}$$

(ii) the interpolating conditions

$$S_j(x_{j-1}) = u_{j-1}, \quad S_j(x_j) = u_j \tag{2.3}$$

(iii) the continuity condition

$$S_j'(x_j^+) = S_j'(x_j^-) \tag{2.4}$$

(iv) the consistency condition

$$\frac{p_j}{2} = tan\frac{p_j}{2}, \quad p_j = h\sqrt{\frac{b_{j-1}+b_j}{2\varepsilon}} \tag{2.5}$$

where,

$$x \in [x_{j-1}, x_j], \quad x_j = jh, \quad j = 1(1)n, \quad h = 1/n.$$

Solving equation (2.2) with the help of (2.3), we obtain

$$S_j(x) = \frac{1}{-sin\ g_j h}\left[A_j\ sin\ g_j(x_{j-1}-x) + B_j\ sin\ g_j(x-x_j)\right] + \frac{\gamma_j}{\beta_j} \tag{2.6}$$

where,

$$A_j = u_j - \tfrac{\gamma_j}{\beta_j}, \quad B_j = u_{j-1} - \tfrac{\gamma_j}{\beta_j}, \quad g_j = \sqrt{\tfrac{\beta_j}{\varepsilon}}, \quad \beta_j = \tfrac{b_{j-1}+b_j}{2}, \quad \gamma_j = \tfrac{f_{j-1}+f_j}{2}.$$

Equation (2.6) together with (2.5) is known as spline in compression ([110]). Using this spline function we will derive the difference scheme in Section 2.3.

For the second case, the approximate solution of the problem (2.1) (when $b(x) \equiv 0$), we seek (as in the first case) as a solution of the differential equation

$$\varepsilon S_j''(x) + \tilde{a}(x)S_j'(x) = \tilde{f}(x) \tag{2.7}$$

whereas, in this case the parameter $p_j$ used in (2.5) will be given by : $p_j = h(a_{j-1}+a_j)/2\varepsilon$.

Solving (2.7) with the help of (2.3), we obtain

$$\begin{aligned} S_j(x) &= \frac{1}{F_j}\left[D_j\ exp\left(-\frac{\alpha_j x_{j-1}}{\varepsilon}\right) - E_j\ exp\left(-\frac{\alpha_j x_j}{\varepsilon}\right)\right] + \frac{E_j - D_j}{F_j}\ exp\left(-\frac{\alpha_j x}{\varepsilon}\right) \\ &\quad + \frac{\gamma_j x}{\alpha_j} - \frac{\gamma_j \varepsilon}{\alpha_j^2} \end{aligned} \tag{2.8}$$

where,

$$F_j = \left[exp\left(-\frac{\alpha_j x_{j-1}}{\varepsilon}\right) - exp\left(-\frac{\alpha_j x_j}{\varepsilon}\right)\right]$$

$$D_j = u_j - \frac{\gamma_j x_j}{\alpha_j} + \frac{\gamma_j \varepsilon}{\alpha_j^2}\ ,\ E_j = u_{j-1} - \frac{\gamma_j x_{j-1}}{\alpha_j} + \frac{\gamma_j \varepsilon}{\alpha_j^2}\ ,\ \alpha_j = \frac{a_{j-1} + a_j}{2}$$

Finally, we consider the most general problem (i.e., when both $a(x)$ and $b(x)$ are non zero).

In this case the corresponding differential equation will be

$$\varepsilon S_j''(x) + \tilde{a}(x) S_j'(x) + \tilde{b}(x) S_j(x) = \tilde{f}(x) \tag{2.9}$$

whereas, in this case we use two parameters $p_j$ and $t_j$ while making use of the consistency condition and these will be given by : $p_j = h\alpha_j/2\varepsilon$ , $t_j = [h(\alpha_j^2 - 4\beta_j\varepsilon)^{1/2}]/2\varepsilon$.

Solving (2.9) with the help of (2.3), we obtain

$$\begin{aligned} S_j(x) &= \frac{1}{-sinh\ g_j h}\ exp\left(-\frac{\alpha_j x}{2\varepsilon}\right) [G_j\ sinh\ g_j(x_{j-1} - x) + H_j\ sinh\ g_j(x - x_j)] \\ &\quad + \frac{\gamma_j}{\beta_j} \end{aligned} \tag{2.10}$$

where,

$$G_j = \left(u_j - \frac{\gamma_j}{\beta_j}\right) exp\left(\frac{\alpha_j x_j}{2\varepsilon}\right),\ \ H_j = \left(u_{j-1} - \frac{\gamma_j}{\beta_j}\right) exp\left(\frac{\alpha_j x_{j-1}}{2\varepsilon}\right),\ \ g_j = \frac{\sqrt{\alpha_j^2 - 4\beta_j\varepsilon}}{2\varepsilon}$$

## 2.3 Derivation of the scheme

Since $S(x) \in C^2[0,1]$, therefore we have

$$S_j'(x_j) = S_{j+1}'(x_j) \tag{2.11}$$

### CASE I

Differentiating (2.6) with respect to $x$, putting $x = x_j$ and using (2.11), we obtain the difference scheme

$$Ru_j = Qf_j\ ,\ j = 1(1)n-1 \tag{2.12}$$

where,

$$Ru_j = r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1}$$

$$Qf_j = q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1}$$

$$u_0 = \alpha_0 \ , \ u_n = \alpha_1$$

$$r_j^- = 1 + \frac{p_j^2}{4}, \ r_j^+ = 1 + \frac{p_{j+1}^2}{4}, \ r_j^c = -4 + r_j^- + r_j^+, \ q_j^- = q_j^+ = \frac{h^2}{4\varepsilon}, \ q_j^c = \frac{h^2}{2\varepsilon} \quad (2.13)$$

where, $p_j$ is defined as in the description of the method for this case.

**CASE II**

Differentiating (2.8) with respect to $x$, putting $x = x_j$ and using (2.11), we obtain the difference scheme given by (2.12), where

$$r_j^- = 1 - \frac{p_j}{2}, \ r_j^+ = 1 + \frac{p_{j+1}}{2}, \ r_j^c = -(r_j^- + r_j^+), \ q_j^- = q_j^+ = \frac{h^2}{4\varepsilon}, \ q_j^c = \frac{h^2}{2\varepsilon} \quad (2.14)$$

where, $p_j$ is defined as in the description of the method for this case.

**CASE III**

Differentiating (2.10) with respect to $x$, putting $x = x_j$ and using (2.11), we obtain the difference scheme given by (2.12), where

$$r_j^- = \left(1 - \frac{t_j^2}{4}\right)\left(\frac{2 - p_j}{2 + p_j}\right), \quad r_j^+ = \left(1 - \frac{t_{j+1}^2}{4}\right)\left(\frac{2 + p_{j+1}}{2 - p_{j+1}}\right)$$

$$r_j^c = -2 + p_j - p_{j+1} - \frac{1}{4}(t_j^2 + t_{j+1}^2)$$

$$q_j^- = \frac{h^2}{2\varepsilon(2 + p_j)}, \ q_j^+ = \frac{h^2}{2\varepsilon(2 - p_{j+1})}, \ q_j^c = q_j^- + q_j^+ \quad (2.15)$$

where, $p_j$ and $t_j$ are defined as in the description of the method for this case.

## 2.4 Proof of the uniform convergence

Throughout the chapter $M$ will denote a positive constant which may take different values in different equations (inequalities) but that are always independent of $h$ and $\varepsilon$.

**CASE I**

The scheme (2.12), (2.13) can be written in the matrix form :

$$Au = F$$

where, A is a matrix of the system (2.12), u and F are corresponding vectors.
Now, the local truncation $\tau_j(\phi)$ of the scheme (2.12), is defined by

$$\tau_j(\phi) = R\phi_j - Q(L\phi)_j$$

where, $\phi(x)$ is an arbitrary sufficiently smooth function. Therefore,

$$\begin{aligned} \tau_j(y) &= Ry_j - Q(Ly)_j \\ &= R(y_j - u_j) \\ \Rightarrow \quad R(y_j - u_j) &= \tau_j(y) \\ \Rightarrow \max_j |y_j - u_j| &\leq ||A^{-1}|| \max_j |\tau_j(y)| \end{aligned} \tag{2.16}$$

In order to estimate the values $|y_j - u_j|$, we will estimate the truncation error $\tau_j(y)$ and the norm of the matrix $A^{-1}$.

The following Lemma [66] gives the properties of the exact solution which are important for the proof:-

**Lemma 2.1** *Let $y(x) \in C^4[0,1]$. Let $b'(0) = b'(1) = 0$. Then the solution of the problem (2.1) (with $a(x) \equiv 0$) has the form :-*

$$y(x) = v(x) + w(x) + g(x)$$

*where,*

$$v(x) = q_0 exp\left[-x\{b(0)/(-\varepsilon)\}^{\frac{1}{2}}\right] \quad , \quad w(x) = q_1 exp\left[-(1-x)\{b(1)/(-\varepsilon)\}^{\frac{1}{2}}\right]$$

*$q_0$ and $q_1$ are bounded functions of $\varepsilon$ independent of $x$ and*

$$|g^{(k)}(x)| \leq N\left(1 + (-\varepsilon)^{1-\frac{k}{2}}\right) \quad , \quad k = 0(1)4$$

*$N$ is a constant independent of $\varepsilon$.*

Estimation of truncation error and the norm of $A^{-1}$:-

From Lemma 2.1, we have

$$\tau_j(y) = \tau_j(v) + \tau_j(w) + \tau_j(g)$$

We will estimate separately the parts of $\tau_j(y)$.

We will start with $v(x)$:-

$$Rv_j = r_j^- v_{j-1} + r_j^c v_j + r_j^+ v_{j+1}$$

Expanding $v_{j-1}$ and $v_{j+1}$ in terms of $v_j$ and using the fact that $v_j$ involves $q_0$ which is a bounded function of $\varepsilon$ independent of $x$ we obtain

$$Rv_j = \frac{h^2}{\varepsilon}(b_j - b_0)v_j + O\left(h^4\right) \tag{2.17}$$

and

$$\begin{aligned} Q(Lv)_j &= q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1} \\ &= q_j^-(\varepsilon v''_{j-1} + b_{j-1}v_{j-1}) + q_j^c(\varepsilon v''_j + b_j v_j) \\ &\quad + q_j^+(\varepsilon v''_{j+1} + b_{j+1}v_{j+1}) \\ &= \frac{h^2}{\varepsilon}(b_j - b_0)v_j + O\left(h^4\right) \end{aligned} \tag{2.18}$$

From equations (2.17) and (2.18), we have

$$|\tau_j(v)| = |Rv_j - Q(Lv)_j| \leq Mh^4 \tag{2.19}$$

Similarly,

$$Rw_j = \frac{h^2}{\varepsilon}(b_j - b_1)w_j + O\left(h^4\right) \tag{2.20}$$

$$Q(Lw)_j = \frac{h^2}{\varepsilon}(b_j - b_1)w_j + O\left(h^4\right) \tag{2.21}$$

$$\Rightarrow \quad |\tau_j(w)| \leq Mh^4 \tag{2.22}$$

Now

$$\tau_j(g) = Rg_j - Q(Lg)_j \tag{2.23}$$

where

$$
\begin{aligned}
Rg_j &= r_j^- g_{j-1} + r_j^c g_j + r_j^+ g_{j+1} \\
&= (r_j^- + r_j^c + r_j^+) g_j + (r_j^+ - r_j^-) h g_j' + (r_j^+ + r_j^-) \frac{h^2}{2!} g_j'' + \cdots
\end{aligned}
$$

and

$$
\begin{aligned}
Q(Lg)_j &= q_j^-(\varepsilon g_{j-1}'' + b_{j-1} g_{j-1}) + q_j^c(\varepsilon g_j'' + b_j g_j) + q_j^+(\varepsilon g_{j+1}'' + b_{j+1} g_{j+1}) \\
&= \left[q_j^- b_{j-1} + q_j^c b_j + q_j^+ b_{j+1}\right] g_j + \left[h(-q_j^- b_{j-1} + q_j^+ b_{j+1})\right] g_j' \\
&\quad + \left[q_j^-(\varepsilon + \frac{h^2}{2} b_{j-1}) + q_j^c \varepsilon + q_j^+(\varepsilon + \frac{h^2}{2} b_{j+1})\right] g_j'' + \cdots
\end{aligned}
$$

Therefore from (2.23), we have

$$\tau_j(g) = T_0 g_j + T_1 g_j' + \text{ Remainder terms}$$

where

$$T_0 = (r_j^- + r_j^c + r_j^+) - (q_j^- b_{j-1} + q_j^c b_j + q_j^+ b_{j+1})$$
$$T_1 = (r_j^+ - r_j^-) h - (q_j^+ b_{j+1} - q_j^- b_{j-1}) h$$

Using (2.13) we see that $T_0 = 0$ and $|T_1| \leq Mh^4/\varepsilon$, therefore

$$
\begin{aligned}
|T_1 g_j'| &\leq (Mh^4/\varepsilon)|g_j'| \\
&\leq Mh^4/\varepsilon \quad \text{(using Lemma 2.1)} \\
\Rightarrow \quad |\tau_j(g)| &\leq Mh^4/\varepsilon \qquad (2.24)
\end{aligned}
$$

Since $h^4 \leq h^4/\varepsilon$, therefore from (2.19), (2.22) and (2.24), we have

$$|\tau_j(y)| \leq Mh^4/\varepsilon \qquad (2.25)$$

Estimate of $\|A^{-1}\|$ :-

Since $r_j^c < 0$ and $r_j^\pm > 0$

therefore,

$$\|A^{-1}\| \leq \max_j |r_j^- + r_j^c + r_j^+|^{-1} \qquad \text{(Varah [248])}$$

Now

$$
\begin{aligned}
|r_j^- + r_j^c + r_j^+| &= \left|\left(1+\frac{p_j^2}{4}\right) + \left(-4+1+\frac{p_{j+1}^2}{4}+1+\frac{p_j^2}{4}\right) + \left(1+\frac{p_{j+1}^2}{4}\right)\right| \\
&= \frac{h^2}{4\varepsilon}\left|b_{j-1}+2b_j+b_{j+1}\right| \\
&\geq M_1\frac{h^2}{\varepsilon} \\
\Rightarrow \qquad |r_j^- + r_j^c + r_j^+|^{-1} &\leq M\frac{\varepsilon}{h^2} \quad \left(\text{where, } M = \frac{1}{M_1}\right) \\
\Rightarrow \max_j |r_j^- + r_j^c + r_j^+|^{-1} &\leq M\frac{\varepsilon}{h^2} \\
\Rightarrow \qquad ||A^{-1}|| &\leq M\frac{\varepsilon}{h^2} \qquad (2.26)
\end{aligned}
$$

Hence from (2.16), (2.25) and (2.26), we have the following theorem:

**Theorem 2.1** *Let $b(x)$, $f(x) \in C^2[0,1]$ and $b(x) \geq b > 0, b'(0) = b'(1) = 0$. Let $u_j$, j=0(1)n, be the approximate solution of (2.1), when $a(x) \equiv 0$, obtained using (2.12), (2.13). Then, there is a constant $C$ independent of $\varepsilon$ and $h$ such that*

$$\max_j |y_j - u_j| \leq Ch^2$$

## CASE II

For the error analysis of the second and third cases we have used the comparison functions method developed by Kellogg and Tsan [131] and Berger et al. [26]. By a comparison function we mean a function $\phi$ such that $L\phi_i > 0$, $-N < i < N$ and $\phi_{\pm N} > 0$, where $L$ is a differential operator and $N$ is a positive integer. These functions are used together with the maximum principle to convert the bounds on truncation error to bounds on descritization error.

This method uses the following two Lemmas [26] :

**Lemma 2.2** *(maximum principle) : Let $\{u_j\}$ be a set of values at the grid points $x_j$, satisfying $u_0 \leq 0, u_n \leq 0$ and $Ru_j \geq 0, j = 1(1)n-1$, then $u_j \leq 0, j = 0(1)n$.*

**Lemma 2.3** *If $K_1(h,\varepsilon) \geq 0$ and $K_2(h,\varepsilon) \geq 0$ are such that*

$$R(K_1(h,\varepsilon)\phi_j + K_2(h,\varepsilon)\psi_j) \geq R(\pm e_j) = \pm\tau_j(y)$$

*for each* $j = 1, 2, \cdots, n-1$, *then the discrete maximum principle implies that*

$$|e_j| \leq K_1(h,\varepsilon)|\phi_j| + K_2(h,\varepsilon)|\psi_j|$$

*where,* $|e_j| = |u_j - y(x_j)|$, *for each* $j$ *and* $\phi$ *and* $\psi$ *are two comparison functions.*

The following Lemma [26] gives the properties of the exact solution of (2.1) (when $b(x) \equiv 0$):

**Lemma 2.4** *Let* $a(x), f(x) \in C^3(0,1)$, *then*

$$y(x) = v(x) + w(x)$$

*where,* $v(x) = \left(-\varepsilon\frac{y'(0)}{a(0)}\right) exp\left(-\frac{a(0)x}{\varepsilon}\right)$ *and* $|w^{(k)}(x)| \leq M\left[1 + \varepsilon^{-k+1} exp(-\frac{\delta x}{\varepsilon})\right]$, $k = 0(1)4$, $\delta = a/4$, *where* $0 < a < a(x)$ *for all* $x$ *and* $M$ *is a positive constant independent of* $h$ *and* $\varepsilon$.

We use two comparison functions (as in [26] ) : $\phi = -2+x$ and $\psi = -exp(-\beta x/\varepsilon)$, ($\beta$ will be taken to be the smallest of various constants appearing in the proof). Therefore $\phi_j = -2 + x_j$ and $\psi_j = -[\mu(\beta)]^j$, $j = 0(1)n$; where, $\mu(\beta) = [r^-(\beta h/\varepsilon)/r^+(\beta h/\varepsilon)] = exp(-\beta h/\varepsilon)$.

Remark :- The following inequalities hold for $h < 2\varepsilon/(a_{j-1} + a_j)$ :

$$R\phi_j \geq Mh^2/\varepsilon \ , \quad R\psi_j \geq M(\mu(\beta))^j h^2/\varepsilon^2$$

Now we estimate the truncation error of the scheme (2.12) using (2.14).

We have,

$$\tau_j(y) = T_0 y_j + T_1 y_j' + T_2 y_j'' + \text{Remainder terms}$$

where,

$$\begin{aligned}
T_0 &= r_j^- + r_j^c + r_j^+ \\
T_1 &= h(r_j^+ - r_j^-) - (q_j^- a_{j-1} + q_j^c a_j + q_j^+ a_{j+1}) \\
T_2 &= \tfrac{h^2}{2}(r_j^- + r_j^+) - \varepsilon(q_j^- + q_j^c + q_j^+) + h(q_j^- a_{j-1} - q_j^+ a_{j+1})
\end{aligned}$$

Using (2.14) we see that $T_0 = 0, T_1 = 0$ and $|T_2| \le Mh^3/\varepsilon$

Therefore using Lemma 2.4, we have

$$\left|T_2 w_j^{(2)}\right| \;\le\; M\frac{h^3}{\varepsilon}\left[1+\frac{1}{\varepsilon}exp\left(-\frac{\delta x_j}{\varepsilon}\right)\right]$$

Also from Lemma 2.4, we have $v_j'' = (-a(0)/\varepsilon)^2 v_j$, therefore

$$|\tau_j(v)| \;\le\; \frac{Mh^3}{\varepsilon^2}exp\left(-\frac{a(0)x_j}{\varepsilon}\right)$$

$$\Rightarrow \quad |\tau_j(y)| \;\le\; M\left[\frac{h^3}{\varepsilon}+\frac{h^3}{\varepsilon^2}exp\left(-\frac{\delta x_j}{\varepsilon}\right)\right]$$

Choosing $K_1 = h^2$ and $K_2 = h^2/\varepsilon$, we see that Lemma 2.3 is satisfied and therefore we have the following theorem:

**Theorem 2.2** *Let $\{u_j\}$, $j = 0(1)n$, be a set of values of the approximate solution to $y(x)$ of (2.1), when $b(x) \equiv 0$, obtained using (2.12) and (2.14). Then there are positive constants $\beta$ and $C$ (independent of $h$ and $\varepsilon$) such that the following estimate holds:*

$$\max_j |y(x_j) - u_j| \;\le\; Ch^2\left[1+\frac{1}{\varepsilon}exp\left(-\frac{\beta x_j}{\varepsilon}\right)\right]$$

## CASE III

We use the same approach as was in CASE II and the same Lemma, i.e., Lemma 2.4 for the properties of the exact solution of (2.1) (as in this case also the boundary layer will be at the left end of the underlying interval).

In this case we observe that $R\phi_j \ge Mh^2/\varepsilon$, $R\psi_j \ge M(\mu(\beta))^j h^2/\varepsilon$, for small $p_j$ and $t_j$ and

$$|\tau_j(y)| \le M\frac{h^2}{\varepsilon}\left[1+exp\left(-\frac{\delta x_j}{\varepsilon}\right)\right]$$

We choose $K_1 = h^2$ and $K_2 = h^2 exp(-\delta x_j/\varepsilon)$ and finally we obtain the following theorem:

**Theorem 2.3** *Let $\{u_j\}$, $j = 0(1)n$, be a set of values of the approximate solution to $y(x)$ of (2.1) obtained using (2.12) and (2.15). Then there are positive constants $\delta$ and $C$ (independent of $h$ and $\varepsilon$) such that the following estimate holds :*

$$\max_j |y(x_j) - u_j| \leq Ch^2 \left[1 + exp\left(\frac{-2\delta x_j}{\varepsilon}\right)\right]$$

## 2.5 Test Examples and Numerical Results

To illustrate the predicted theory, we solve the following problems :

**Example 2.1** [24] : *Consider*

$$\varepsilon y'' + y = 0 \;\; ; \;\; y(0) = 0 \;\; , \;\; y(1) = 1$$

*whose exact solution is given by*

$$y(x) = \frac{sin(x/\sqrt{\varepsilon})}{sin(1/\sqrt{\varepsilon})} \;\; , \;\; \varepsilon \neq (n\pi)^{-2}$$

**Example 2.2** [135] : *Consider*

$$\varepsilon y'' + y' = 2 \;\; ; \;\; y(0) = 0 \;\; , \;\; y(1) = 1$$

*whose exact solution is given by*

$$y(x) = 2x + \frac{1 - e^{-(x/\varepsilon)}}{e^{-(1/\varepsilon)} - 1}$$

**Example 2.3** [201] : *Consider*

$$\varepsilon y'' + y' + y = 0 \;\; ; \;\; y(0) = 1 \;\; , \;\; y(1) = 2$$

*whose exact solution is given by*

$$y(x) = \frac{[(2 - e^{r_2})e^{r_1 x} + (e^{r_1} - 2)e^{r_2 x}]}{(e^{r_1} - e^{r_2})}$$

*where,*

$$r_1 = \frac{-1 + \sqrt{1 - 4\varepsilon}}{2\varepsilon} \;\; , \;\; r_2 = \frac{-1 - \sqrt{1 - 4\varepsilon}}{2\varepsilon}$$

**Example 2.4** *[209] : Consider*

$$\varepsilon y'' + (x+1)^3 y' = f(x) \ ; \ y(0) = 2 \ , \ y(1) = \left(\frac{1}{8}\right) exp\left(-\frac{15}{4\varepsilon}\right) + exp\left(-\frac{1}{2}\right)$$

*whose exact solution is given by*

$$y(x) = \frac{1}{(x+1)^3} exp\left[-\frac{1}{4\varepsilon}\{(x+1)^4 - 1\}\right] + exp\left(-\frac{x}{2}\right)$$

**Example 2.5** *[80] : Consider*

$$\varepsilon y'' + (1+x)^2 y' + 2(1+x)y = exp\left(-\frac{x}{2}\right)\left[(1+x)(3-x) + \left(\frac{\varepsilon}{2}\right)\right]/2$$

$$y(0) = 0 \ , \ y(1) = exp\left(-\frac{1}{2}\right) - exp\left(-\frac{7}{3\varepsilon}\right) .$$

*whose exact solution is given by*

$$y(x) = exp\left(-\frac{x}{2}\right) - exp\left[-\frac{\{x(x^2+3x+3)\}}{3\varepsilon}\right]$$

Tables 2.1, 2.3, 2.5, 2.7 and 2.9 contain the maximum errors at all the mesh points:

$$\max_j |y(x_j) - u_j| \tag{2.27}$$

for different $n$ and $\varepsilon$, $n = 1/h$ .

Tables 2.2, 2.4, 2.6, 2.8 and 2.10 contain the numerical rate of uniform convergence which is determined as in [66]:

$$r_{k,\varepsilon} = log_2\left(z_{k,\varepsilon}/z_{k+1,\varepsilon}\right) \ , \ k = 0, 1, \cdots\cdots\cdots \tag{2.28}$$

where,

$$z_{k,\varepsilon} = \max_j |u_j^{h/2^k} - u_{2j}^{h/2^{k+1}}| \ , \ k = 0, 1, \cdots\cdots\cdots$$

and $u_j^{h/2^k}$ denotes the value of $u_j$ for the mesh length $h/2^k$.

Table 2.1: Max. Errors for Example 2.1

| $\varepsilon$ | n = 32 | n = 64 | n = 128 | n = 256 | n = 512 | n = 1024 | n = 2048 | n = 4096 |
|---|---|---|---|---|---|---|---|---|
| 1/4 | 0.47E-03 | 0.12E-03 | 0.29E-04 | 0.74E-05 | 0.18E-05 | 0.46E-06 | 0.12E-06 | 0.29E-07 |
| 1/8 | 0.19E-01 | 0.47E-02 | 0.12E-02 | 0.29E-03 | 0.73E-04 | 0.18E-04 | 0.46E-05 | 0.11E-05 |
| 1/16 | 0.71E-02 | 0.18E-02 | 0.44E-03 | 0.11E-03 | 0.28E-04 | 0.69E-05 | 0.17E-05 | 0.43E-06 |
| 1/32 | 0.38E-01 | 0.98E-02 | 0.25E-02 | 0.62E-03 | 0.15E-03 | 0.39E-04 | 0.96E-05 | 0.24E-05 |
| 1/64 | 0.34E-01 | 0.87E-02 | 0.22E-02 | 0.55E-03 | 0.14E-03 | 0.34E-04 | 0.85E-05 | 0.21E-05 |
| 1/128 | —— | 0.28E-01 | 0.72E-02 | 0.18E-02 | 0.45E-03 | 0.11E-03 | 0.28E-04 | 0.70E-05 |
| 1/256 | —— | —— | —— | 0.63E-01 | 0.16E-01 | 0.39E-02 | 0.97E-03 | 0.24E-03 |
| 1/512 | —— | —— | —— | 0.42E-01 | 0.10E-01 | 0.25E-02 | 0.64E-03 | 0.16E-03 |

Table 2.2: Rate of Convergence for Example 2.1
n = 128, 256, 512, 1024, 2048

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/8 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/16 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/32 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/64 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/128 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/256 | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/512 | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |

Table 2.3: Max. Errors for Example 2.2

| $\varepsilon$ | n=32 | n=64 | n= 128 | n=256 | n=512 | n=1024 | n=2048 | n=4096 |
|---|---|---|---|---|---|---|---|---|
| 1/4 | 0.43E-03 | 0.11E-03 | 0.27E-04 | 0.67E-05 | 0.17E-05 | 0.42E-06 | 0.10E-06 | 0.26E-07 |
| 1/8 | 0.19E-02 | 0.48E-03 | 0.12E-03 | 0.30E-04 | 0.75E-05 | 0.19E-05 | 0.47E-06 | 0.12E-06 |
| 1/16 | 0.79E-02 | 0.19E-02 | 0.48E-03 | 0.12E-03 | 0.30E-04 | 0.75E-05 | 0.19E-05 | 0.47E-06 |
| 1/32 | 0.35E-01 | 0.79E-02 | 0.19E-02 | 0.48E-03 | 0.12E-03 | 0.30E-04 | 0.75E-05 | 0.19E-05 |
| 1/64 | —— | 0.35E-01 | 0.79E-02 | 0.19E-02 | 0.48E-03 | 0.12E-03 | 0.30E-04 | 0.75E-05 |
| 1/128 | —— | —— | 0.35E-01 | 0.79E-02 | 0.19E-02 | 0.48E-03 | 0.12E-03 | 0.30E-04 |
| 1/256 | —— | —— | —— | 0.35E-01 | 0.79E-02 | 0.19E-02 | 0.48E-03 | 0.12E-03 |
| 1/512 | —— | —— | —— | —— | 0.35E-01 | 0.79E-02 | 0.19E-02 | 0.48E-03 |

Table 2.4: Rate of Convergence for Example 2.2
n = 128, 256, 512, 1024, 2048

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/8 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/16 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/32 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/64 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/128 | 0.22E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/256 | 0.21E+01 | 0.22E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.21E+01 |
| 1/512 | —— | 0.21E+01 | 0.22E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |

Table 2.5: Max. Errors for Example 2.3

| $\varepsilon$ | n= 64 | n=128 | n=256 | n=512 | n=1024 | n=2048 | n=4096 | n=8192 |
|---|---|---|---|---|---|---|---|---|
| 1/8 | 0.25E-02 | 0.64E-03 | 0.16E-03 | 0.40E-04 | 0.99E-05 | 0.25E-05 | 0.62E-06 | 0.16E-06 |
| 1/16 | 0.15E-01 | 0.37E-02 | 0.91E-03 | 0.23E-03 | 0.57E-04 | 0.14E-04 | 0.36E-05 | 0.89E-06 |
| 1/32 | 0.71E-01 | 0.17E-01 | 0.44E-02 | 0.11E-02 | 0.27E-03 | 0.68E-04 | 0.17E-04 | 0.42E-05 |
| 1/64 | —— | 0.78E-01 | 0.19E-01 | 0.48E-02 | 0.12E-02 | 0.30E-03 | 0.75E-04 | 0.19E-04 |
| 1/128 | —— | —— | 0.82E-01 | 0.20E-01 | 0.50E-02 | 0.13E-02 | 0.31E-03 | 0.78E-04 |
| 1/256 | —— | —— | —— | 0.84E-01 | 0.21E-01 | 0.52E-02 | 0.13E-02 | 0.32E-03 |
| 1/512 | —— | —— | —— | —— | 0.86E-01 | 0.21E-01 | 0.52E-02 | 0.13E-02 |
| 1/1024 | —— | —— | —— | —— | —— | 0.86E-01 | 0.21E-01 | 0.53E-02 |

Table 2.6: Rate of Convergence for Example 2.3
n = 256, 512, 1024, 2048, 4096

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/8 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/16 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/32 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/64 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/128 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/256 | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/512 | —— | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |

Table 2.7: Max. Errors for Example 2.4

| $\varepsilon$ | n=32 | n=64 | n=128 | n=256 | n=512 | n=1024 | n=2048 | n=4096 |
|---|---|---|---|---|---|---|---|---|
| 1/4 | 0.16E-02 | 0.40E-03 | 0.99E-04 | 0.25E-04 | 0.62E-05 | 0.15E-05 | 0.39E-06 | 0.97E-07 |
| 1/8 | 0.32E-02 | 0.80E-03 | 0.20E-03 | 0.50E-04 | 0.12E-04 | 0.31E-05 | 0.78E-06 | 0.19E-06 |
| 1/16 | 0.94E-02 | 0.23E-02 | 0.57E-03 | 0.14E-03 | 0.36E-04 | 0.89E-05 | 0.22E-05 | 0.56E-06 |
| 1/32 | 0.37E-01 | 0.83E-02 | 0.20E-02 | 0.51E-03 | 0.13E-03 | 0.32E-04 | 0.79E-05 | 0.20E-05 |
| 1/64 | —— | 0.35E-01 | 0.80E-02 | 0.20E-02 | 0.49E-03 | 0.12E-03 | 0.30E-04 | 0.76E-05 |
| 1/128 | —— | —— | 0.35E-01 | 0.79E-02 | 0.19E-02 | 0.48E-03 | 0.12E-03 | 0.30E-04 |
| 1/256 | —— | —— | —— | 0.35E-01 | 0.79E-02 | 0.19E-02 | 0.48E-03 | 0.12E-03 |
| 1/512 | —— | —— | —— | —— | 0.35E-01 | 0.79E-02 | 0.19E-02 | 0.48E-03 |

Table 2.8: Rate of Convergence for Example 2.4
n = 128, 256, 512, 1024, 2048

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/8 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/16 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/32 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/64 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/128 | 0.22E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/256 | 0.21E+01 | 0.22E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.21E+01 |
| 1/512 | —— | 0.21E+01 | 0.22E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |

Table 2.9: Max. Errors for Example 2.5

| $\varepsilon$ | n=64 | n=128 | n=256 | n=512 | n=1024 | n=2048 | n=4096 | n=8192 |
|---|---|---|---|---|---|---|---|---|
| 1/8 | 0.36E-02 | 0.89E-03 | 0.22E-03 | 0.56E-04 | 0.14E-04 | 0.35E-05 | 0.87E-06 | 0.22E-06 |
| 1/16 | 0.18E-01 | 0.44E-02 | 0.11E-02 | 0.27E-03 | 0.68E-04 | 0.17E-04 | 0.43E-05 | 0.11E-05 |
| 1/32 | —— | 0.21E-01 | 0.50E-02 | 0.12E-02 | 0.31E-03 | 0.78E-04 | 0.19E-04 | 0.49E-05 |
| 1/64 | —— | —— | 0.23E-01 | 0.55E-02 | 0.14E-02 | 0.34E-03 | 0.84E-04 | 0.21E-04 |
| 1/128 | —— | —— | —— | 0.24E-01 | 0.58E-02 | 0.14E-02 | 0.36E-03 | 0.89E-04 |
| 1/256 | —— | —— | —— | —— | 0.25E-01 | 0.59E-02 | 0.15E-02 | 0.37E-03 |
| 1/512 | —— | —— | —— | —— | —— | 0.25E-01 | 0.60E-02 | 0.15E-02 |
| 1/1024 | —— | —— | —— | —— | —— | —— | 0.26E-01 | 0.61E-02 |

Table 2.10: Rate of Convergence for Example 2.5
n = 256, 512, 1024, 2048, 4096

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/8 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/16 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/32 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/64 | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/128 | 0.23E+01 | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.21E+01 |
| 1/256 | —— | 0.23E+01 | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.21E+01 |
| 1/512 | —— | —— | 0.23E+01 | 0.21E+01 | 0.20E+01 | 0.21E+01 |

## 2.6 Discussion

In this chapter we have described the numerical method for solving singular perturbation problems using spline in compression. It is a practical method and can easily be implemented on a computer to solve such problems. The method has been analysed for convergence. Test examples have been solved to demonstrate the efficiency of the proposed method.

The consistency condition needs the parameter(s) to be small. Therefore in the tables, we included the results corresponding to only those values of $h$ and $\varepsilon$ which satisfy this requirement.

Tables 2.2, 2.4, 2.6, 2.8 and 2.10 show the rate of uniform convergence as 2 for the problems of the type which are considered in the respective cases.

The problems of the type as considered in the second case have earlier been solved by Stojanovic [222] and Surla and Uzelac [237] for the same values of $h$ and $\varepsilon$ as what we have taken but they could get only first order of uniform convergence, whereas the present method has second order of uniform convergence. Example 2.3 and example 2.4 have also been solved by Roberts [201] and Sakai and Usmani [209] respectively. Their method works for more smaller values of $\varepsilon$ but they too did not achieve uniform convergence. However our method works for more smaller values of $\varepsilon$ also but in that case we require $h$ also to be very small so as to satisfy the above mentioned requirement.

As the parameter (used in the consistency condition) increases, the error also increases. This can be observed from the respective columns of the Tables 2.1, 2.3, 2.5, 2.7 and 2.9

Finally we would like to remark that the solution of the equations considered in the first case is having oscillatory behaviour (Bender and Orszag [24] and Flaherty and Mathon [80]). Also $r_j's$ and $q_j's$ in the schemes for case II and case III without using the consistency condition, involves $exp(-\kappa(h,\varepsilon))$ and $exp(+\kappa(h,\varepsilon))$ terms, where $\kappa(h,\varepsilon)$ is a function of $h$ and $\varepsilon$. Therefore in these cases the $r_j's$ and $q_j's$ will tend to 0 and(or) $\infty$. Hence the system thus obtained may not be well behaved. To overcome spurious oscillations in the solution of the equation considered in the first case as well as the above mentioned difficulties in the second and the third cases, we use the consistency condition.

# Chapter 3

## SPLINE IN TENSION FOR SINGULAR PERTURBATION PROBLEMS

## 3.1 Introduction

In this chapter we consider the following class of singularly perturbed two point boundary value problems

$$\left.\begin{array}{rl} Ly \equiv & -\ \varepsilon y'' + a(x)y' + b(x)y = f(x) \quad \text{on} \quad (0,1) \\ & y(0) = \alpha_0, \ y(1) = \alpha_1, \ \alpha_0, \alpha_1 \in R \end{array}\right\} \tag{3.1}$$

where, $x \in (0,1)$, $\varepsilon$ is a small positive parameter and $a(x)$, $b(x)$ and $f(x)$ are bounded continuous functions. In many applications, (3.1) possesses boundary or interior layers, i.e. regions of rapid change in the solution near the endpoints or some interior points, with width $O(1)$ as $\varepsilon \to 0$. We assume that $a(x) \geq a > 0$ on $[0,1]$. This condition prohibits the development of turning points on interior layers [182].

The problem (3.1) models many practical problems in various areas of science and engineering, e.g., fluid mechanics, quantum mechanics, optimal control, chemical-reactor theory, aerodynamics, reaction-diffusion process, geophysics etc.

Out of the three principal numerical approaches to solve the model problems like (3.1), namely, the Finite Difference Methods, the Finite-Element Methods and the Spline Approximation Methods, The first two have been used by numerous researchers. Abrahamsson et al [1] and Kreiss & Kreiss [140] used finite difference methods whereas Chin and Krasny [51] used finite element methods to solve such problems.

The present chapter is concerned with the third one, namely, the Spline Approximation Method, to solve the problems of the type (3.1). It is known that the most classical methods fail when $\varepsilon$ is small relative to the mesh width $h$ that is used for discretization

of the operator $L$. Our aim is to show that tension splines can furnish accurate numerical approximations of (3.1) when $\varepsilon$ is either small or large as compared to $h$. Tension splines were first used by Schweikert [216] as a means of eliminating spurious oscillations in curve fitting with cubic splines. They have been subsequently studied by Pruess [197], de Boor [54] and others.

In this chapter we have shown that by making use of the continuity of the first derivative of the spline function, the resulting spline difference scheme gives a tridiagonal system which can be solved efficiently by the well known algorithms. We consider three types of problems. First we analyse the problems in which the second derivative term and the function term are present while the term containing the first derivative is absent. The problems having the second and first derivative terms but lacking the function term are considered in the second case. Finally the third case deals with the most general problem.

In Section 3.2 we give a brief description of the method. The derivation of the difference schemes for all the three cases has been given in Section 3.3. In Section 3.4 we have shown the second order accuracy of the method. In Section 3.5 we have solved six numerical Examples to demonstrate the applicability of the method. The discussion on our results is given in Section 3.6.

## 3.2 Description of the Method

For $x \in [x_{j-1}, x_j]$, we define $\widetilde{a}(x) = (a_{j-1} + a_j)/2$ and anologously $\widetilde{b}(x)$ and $\widetilde{f}(x)$ too. Consider first the equation (3.1), with $a(x) \equiv 0$.

The approximate solution of this problem, is sought in the form of the function S(x), which on each interval $[x_{j-1}, x_j]$, (denoted by $S_j(x)$) satisfies the following relations :-

(i) the differential equation

$$- \varepsilon S_j''(x) + \widetilde{b}(x) S_j(x) = \widetilde{f}(x) \tag{3.2}$$

(ii) the interpolating conditions

$$S_j(x_{j-1}) = u_{j-1}, \quad S_j(x_j) = u_j \tag{3.3}$$

(iii) the continuity condition

$$S_j'(x_j^+) = S_j'(x_j^-) \tag{3.4}$$

where,

$$x \in [x_{j-1}, x_j], \quad x_j = jh, \quad j = 1(1)n, \quad h = 1/n.$$

Solving equation (3.2) with the help of (3.3), we obtain

$$S_j(x) = \frac{1}{-sinh\ g_j h}\left[A_j\ sinh\ g_j(x_{j-1} - x) + B_j\ sinh\ g_j(x - x_j)\right] + \frac{\gamma_j}{\beta_j} \tag{3.5}$$

where,

$$A_j = u_j - \frac{\gamma_j}{\beta_j},\quad B_j = u_{j-1} - \frac{\gamma_j}{\beta_j},\quad g_j = \sqrt{\frac{\beta_j}{\varepsilon}},\quad \beta_j = \frac{b_{j-1} + b_j}{2},\quad \gamma_j = \frac{f_{j-1} + f_j}{2}$$

Equation (3.5) is known as spline in tension [197]. Using this spline function we will derive the difference scheme in Section 3.3.

For the second case, the approximate solution of the problem (3.1) (when $b(x) \equiv 0$), we seek (as in the first case) as a solution of the differential equation

$$-\ \varepsilon S_j''(x) + \tilde{a}(x) S_j'(x) = \tilde{f}(x) \tag{3.6}$$

Solving (3.6) with the help of (3.3), we obtain

$$\begin{aligned} S_j(x) &= \frac{1}{F_j}\left[D_j\ exp\left(\frac{\alpha_j x_{j-1}}{\varepsilon}\right) - E_j\ exp\left(\frac{\alpha_j x_j}{\varepsilon}\right)\right] \\ &\quad + \frac{E_j - D_j}{F_j}\ exp\left(\frac{\alpha_j x}{\varepsilon}\right) + \frac{\gamma_j x}{\alpha_j} + \frac{\gamma_j \varepsilon}{\alpha_j^2} \end{aligned} \tag{3.7}$$

where,

$$F_j = \left[exp\left(\frac{\alpha_j x_{j-1}}{\varepsilon}\right) - exp\left(\frac{\alpha_j x_j}{\varepsilon}\right)\right]$$

$$D_j = u_j - \frac{\gamma_j x_j}{\alpha_j} - \frac{\gamma_j \varepsilon}{\alpha_j^2},\quad E_j = u_{j-1} - \frac{\gamma_j x_{j-1}}{\alpha_j} - \frac{\gamma_j \varepsilon}{\alpha_j^2},\quad \alpha_j = \frac{a_{j-1} + a_j}{2}$$

Finally, we consider the most general problem (i.e., when both $a(x)$ and $b(x)$ are non zero).

In this case the corresponding differential equation will be :

$$-\ \varepsilon S_j''(x) + \tilde{a}(x) S_j'(x) + \tilde{b}(x) b S_j(x) = \tilde{f}(x) \tag{3.8}$$

Solving (3.8) with the help of (3.3), we obtain

$$\begin{aligned} S_j(x) &= \frac{1}{-sinh\ g_j h} exp\left(\frac{\alpha_j x}{2\varepsilon}\right)[G_j\ sinh\ g_j(x_{j-1}-x) + H_j\ sinh\ g_j(x-x_j)] \\ &+\frac{\gamma_j}{\beta_j} \end{aligned} \tag{3.9}$$

where,

$$G_j = \left(u_j - \frac{\gamma_j}{\beta_j}\right) exp\left(-\frac{\alpha_j x_j}{2\varepsilon}\right),\ H_j = \left(u_{j-1} - \frac{\gamma_j}{\beta_j}\right) exp\left(-\frac{\alpha_j x_{j-1}}{2\varepsilon}\right)$$

$$g_j = \frac{\sqrt{\alpha_j^2 + 4\beta_j\varepsilon}}{2\varepsilon}$$

## 3.3 Derivation of the scheme

Since $S(x) \in C^2[0,1]$, therefore we have

$$S_j'(x_j) = S_{j+1}'(x_j) \tag{3.10}$$

**CASE I**

Differentiating (3.5) with respect to $x$, putting $x = x_j$ and using (3.10), we obtain the difference scheme

$$Ru_j = Qf_j\ ,\ j = 1(1)n-1 \tag{3.11}$$

where,

$$Ru_j = r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1}$$

$$Qf_j = q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1}$$

$$u_0 = \alpha_0\ ,\ u_n = \alpha_1$$

$$r_j^- = \frac{p_j}{sinh\ p_j}\ ,\ r_j^+ = \frac{p_{j+1}}{sinh\ p_{j+1}}\ ,\ r_j^c = -(p_j\ coth\ p_j + p_{j+1}\ coth\ p_{j+1})$$

$$q_j^- = -\frac{p_j}{2\beta_j}\left(coth\ p_j - \frac{1}{sinh\ p_j}\right)\ ,\ q_j^+ = -\frac{p_{j+1}}{2\beta_{j+1}}\left(coth\ p_{j+1} - \frac{1}{sinh\ p_{j+1}}\right)$$

$$q_j^c = q_j^- + q_j^+ \tag{3.12}$$

where,

$$p_j = h\sqrt{\frac{\beta_j}{\varepsilon}}$$

**CASE II**

Differentiating (3.7) with respect to $x$, putting $x = x_j$ and using (3.10), we obtain the difference scheme given by (3.11), where

$$r_j^- = \frac{\alpha_j/\varepsilon}{exp(-\alpha_j h/\varepsilon) - 1} \, , \; r_j^+ = \frac{\alpha_{j+1}/\varepsilon}{1 - exp(\alpha_{j+1}h/\varepsilon)} \, , \; r_j^c = -(r_j^- + r_j^+)$$

$$q_j^- = -\frac{1}{2\alpha_j}(1 + hr_j^-) \, , \; q_j^+ = \frac{1}{2\alpha_{j+1}}(1 + hr_j^+)$$

$$q_j^c = q_j^- + q_j^+ \tag{3.13}$$

**CASE III**

Differentiating (3.9) with respect to $x$, putting $x = x_j$ and using (3.10), we obtain the difference scheme given by (3.11), where

$$r_j^- = \left(\frac{p_j}{sinh\, p_j}\right) exp(t_j) \, , \; r_j^+ = \left(\frac{p_{j+1}}{sinh\, p_{j+1}}\right) exp(-t_{j+1})$$

$$r_j^c = -p_j \, coth\, p_j - p_{j+1} \, coth\, p_{j+1} + t_{j+1} - t_j$$

$$q_j^- = \frac{1}{2\beta_j}(r_j^- - t_j - p_j \, coth\, p_j) \, , \; q_j^+ = \frac{1}{2\beta_{j+1}}(r_j^+ + t_{j+1} - p_{j+1} \, coth\, p_{j+1})$$

$$q_j^c = q_j^- + q_j^+ \tag{3.14}$$

where,

$$p_j = \frac{h\sqrt{\alpha_j^2 + 4\beta_j\varepsilon}}{2\varepsilon} \quad \text{and} \quad t_j = \frac{\alpha_j h}{2\varepsilon}$$

## 3.4 Proof of the uniform convergence

Throughout the chapter $M$ will denote a positive constant which may take different values in different equations (inequalities) but that are always independent of $h$ and $\varepsilon$.

**CASE I**

The scheme (3.11), (3.12) can be written in the matrix form :

$$Au = F$$

where, A is a matrix of the system (3.11), u and F are corresponding vectors.
Now, the local truncation $\tau_j(\phi)$ of the scheme (3.11), is defined by

$$\tau_j(\phi) = R\phi_j - Q(L\phi)_j$$

where, $\phi(x)$ is an arbitrary sufficiently smooth function.
Therefore,

$$\begin{aligned} \tau_j(y) &= Ry_j - Q(Ly)_j \\ &= R(y_j - u_j) \\ \Rightarrow \quad R(y_j - u_j) &= \tau_j(y) \\ \Rightarrow \max_j |y_j - u_j| &\leq \|A^{-1}\| \max_j |\tau_j(y)| \end{aligned} \tag{3.15}$$

In order to estimate the values $|y_j - u_j|$, we will estimate the truncation error $\tau_j(y)$ and the norm of the matrix $A^{-1}$.

The following Lemma gives the properties of the exact solution which are important for the proof:-

**Lemma 3.1** *Let $y(x) \in C^4[0,1]$. Let $b'(0) = b'(1) = 0$. Then the solution of the problem (3.1), with $a(x) \equiv 0$, has the form :-*

$$y(x) = v(x) + w(x) + g(x)$$

*where,*

$$v(x) = q_0 exp\left[-x\{b(0)/\varepsilon\}^{\frac{1}{2}}\right] \quad , \quad w(x) = q_1 exp\left[-(1-x)\{b(1)/\varepsilon\}^{\frac{1}{2}}\right]$$

*$q_0$ and $q_1$ are bounded functions of $\varepsilon$ independent of $x$ and*

$$\left|g^{(k)}(x)\right| \leq N\left(1 + \varepsilon^{1-\frac{k}{2}}\right) \quad , \quad k = 0(1)4$$

*$N$ is a constant independent of $\varepsilon$.*

<u>Estimation of truncation error and the norm of $A^{-1}$:-</u>
From Lemma 3.1, we have

$$\tau_j(y) = \tau_j(v) + \tau_j(w) + \tau_j(g)$$

We will estimate separately the parts of $\tau_j(y)$.

First we consider the case in which $h^2 \le \varepsilon$:

We will start with $v(x)$:-

$$Rv_j \;=\; r_j^- v_{j-1} + r_j^c v_j + r_j^+ v_{j+1} \tag{3.16}$$

Expanding $v_{j-1}$ and $v_{j+1}$ in terms of $v_j$, we obtain

$$Rv_j = v_j \left[ r_j^- exp(h\sqrt{b_0/\varepsilon}) + r_j^c + r_j^+ exp(-h\sqrt{b_0/\varepsilon}) \right]$$

Expanding exponentials and using (3.12) with

$$\frac{x}{sinh\ x} = 1 - \frac{x^2}{6} + O(x^4)$$

and

$$x\ coth\ x = 1 + \frac{x^2}{3} + O(x^4)$$

and the fact that $v_j$ involves $q_0$ which is a bounded function of $\varepsilon$ independent of $x$, we obtain

$$Rv_j \;=\; \frac{h^2}{\varepsilon}(b_0 - b_j)v_j + O\left(h^4\right) \tag{3.17}$$

and

$$\begin{aligned} Q(Lv)_j \;&=\; q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1} \\ &=\; q_j^-(-\varepsilon v_{j-1}'' + b_{j-1}v_{j-1}) + q_j^c(-\varepsilon v_j'' + b_j v_j) \\ &\quad + q_j^+(-\varepsilon v_{j+1}'' + b_{j+1}v_{j+1}) \end{aligned} \tag{3.18}$$

Now from Lemma 3.1, we have

$$v(x) \;=\; q_0\ exp\left[ -x \left\{ \frac{b(0)}{\varepsilon} \right\}^{\frac{1}{2}} \right]$$

$$\Rightarrow \quad v_{j-1} = v_j\ exp\left( h\sqrt{\frac{b_0}{\varepsilon}} \right) \quad , \quad v_{j+1} = v_j\ exp\left( -h\sqrt{\frac{b_0}{\varepsilon}} \right)$$

$$v_{j-1}'' = \left( \frac{b_0}{\varepsilon} \right)\ v_{j-1}\ ,\ \ v_j'' = \left( \frac{b_0}{\varepsilon} \right)\ v_j \ \text{ and }\ v_{j+1}'' = \left( \frac{b_0}{\varepsilon} \right)\ v_{j+1}$$

Putting all these expressions into (3.18), we obtain

$$Q(Lv)_j = \frac{h^2}{\varepsilon}(b_0 - b_j)v_j + O\left(h^4\right) \tag{3.19}$$

From equations (3.17) and (3.19), we have

$$|\tau_j(v)| = |Rv_j - Q(Lv)_j| \le Mh^4 \tag{3.20}$$

Similarly,

$$\begin{aligned} Rw_j &= \frac{h^2}{\varepsilon}(b_1 - b_j)w_j + O\left(h^4\right) && (3.21)\\ Q(Lw)_j &= \frac{h^2}{\varepsilon}(b_1 - b_j)w_j + O\left(h^4\right) && (3.22)\\ \Rightarrow \quad |\tau_j(w)| &\le Mh^4 && (3.23) \end{aligned}$$

Now

$$\tau_j(g) = Rg_j - Q(Lg)_j \tag{3.24}$$

where

$$\begin{aligned} Rg_j &= r_j^- g_{j-1} + r_j^c g_j + r_j^+ g_{j+1} \\ &= (r_j^- + r_j^c + r_j^+)g_j + (r_j^+ - r_j^-)hg_j' + (r_j^+ + r_j^-)\frac{h^2}{2!}g_j'' + \cdots \end{aligned}$$

and

$$\begin{aligned} Q(Lg)_j &= q_j^-(-\varepsilon g_{j-1}'' + b_{j-1}g_{j-1}) + q_j^c(-\varepsilon g_j'' + b_j g_j) + q_j^+(-\varepsilon g_{j+1}'' + b_{j+1}g_{j+1}) \\ &= \left[q_j^- b_{j-1} + q_j^c b_j + q_j^+ b_{j+1}\right] g_j + \left[h\left(-q_j^- b_{j-1} + q_j^+ b_{j+1}\right)\right] g_j' \\ &\quad + \left[q_j^-\left(-\varepsilon + \frac{h^2}{2}b_{j-1}\right) - q_j^c\varepsilon + q_j^+\left(-\varepsilon + \frac{h^2}{2}b_{j+1}\right)\right] g_j'' + \cdots \end{aligned}$$

Therefore from (3.24), we have

$$\tau_j(g) = T_0 g_j + T_1 g_j' + \text{ Remainder terms}$$

where

$$T_0 = (r_j^- + r_j^c + r_j^+) - (q_j^- b_{j-1} + q_j^c b_j + q_j^+ b_{j+1})$$
$$T_1 = (r_j^+ - r_j^-)h - (q_j^+ b_{j+1} - q_j^- b_{j-1})h$$

Using (3.12) we see that $T_0 = 0$ and $|T_1| \leq Mh^4/\varepsilon$, therefore

$$\begin{aligned} |T_1 g_j'| &\leq (Mh^4/\varepsilon)|g_j'| \\ &\leq Mh^4/\varepsilon \quad \text{(using Lemma 3.1)} \\ \Rightarrow \quad |\tau_j(g)| &\leq Mh^4/\varepsilon \end{aligned} \tag{3.25}$$

From (3.20), (3.23) and (3.25), we have

$$|\tau_j(y)| \leq Mh^4/\varepsilon \tag{3.26}$$

Now we consider the case in which $h^2 \geq \varepsilon$:

We introduce the notations $r_j^- = r_j^-(p_{j-1})$, $r_j^+ = r_j^+(p_{j+1})$ and $r_j^c = r_j^c(p_j)$. Putting $p_{j-1} = p_{j+1} = p_j = p_0$ in $Rv_j$, we obtain $Rv_j = 0$. Denoting this expression by $\widetilde{R}v_j$. Thus

$$\begin{aligned} Rv_j &= Rv_j - \widetilde{R}v_j \\ &= \left[r_j^-(p_{j-1}) - r_j^-(p_0)\right] v_{j-1} + \left[r_j^c(p_j) - r_j^c(p_0)\right] v_j \\ &\quad + \left[r_j^+(p_{j+1}) - r_j^+(p_0)\right] v_{j+1} \end{aligned}$$

Since

$$|\beta_{j-1} - \beta_0| \leq Mx_{j-1}^2 \;\; , \;\; |\beta_j - \beta_0| \leq Mx_j^2 \;\; , \;\; |\beta_{j+1} - \beta_0| \leq Mx_{j+1}^2$$

Therefore

$$\left|r_j^-(p_{j-1}) - r_j^-(p_0)\right| \leq Mx_{j-1}^2 h^2/\varepsilon \;\; , \;\; \left|r_j^+(p_{j+1}) - r_j^+(p_0)\right| \leq Mx_{j+1}^2 h^2/\varepsilon$$

$$\left|r_j^c(p_j) - r_j^c(p_0)\right| \leq M\left(x_j^2 + x_{j+1}^2\right) h^2/\varepsilon$$

Therefore

$$|Rv_j| \leq M\frac{h^2}{\varepsilon}\left[x_{j-1}^2 v_{j-1} + \left(x_j^2 + x_{j+1}^2\right) v_j + x_{j+1}^2 v_{j+1}\right]$$

Now using the fact that (see, e.g., Doolan et al. [66])

$$x\, exp\left(-\frac{cx}{\varepsilon}\right) \leq M\left(\frac{\varepsilon}{c}\right) exp\left(-\frac{cx}{2\varepsilon}\right)$$

and $v_{j's}$ involve $q_0$ which is a bounded function of $\varepsilon$, we obtain

$$x_{j-1}^2 v_{j-1} \le Mh^2\varepsilon \ , \ x_j^2 v_j \le Mh^2\varepsilon \ , \ x_{j+1}^2 v_{j+1} \le Mh^2\varepsilon \text{ and } x_{j+1}^2 v_j \le Mh^2\varepsilon$$

Hence

$$|Rv_j| \le Mh^4$$

Now

$$Q(Lv)_j = q_j^- v_{j-1}(b_{j-1} - b_0) + q_j^c v_j (b_j - b_0) + q_j^+ v_{j+1}(b_{j+1} - b_0)$$

But

$$\left|q^{\pm c}\right| \le Mh^2/\varepsilon \text{ and } |b_{j-1} - b_0| \le Mx_{j-1}^2 \text{, etc.}$$

Therefore

$$|Q(Lv)_j| \ \le \ M\frac{h^2}{\varepsilon}\left[x_{j-1}^2 v_{j-1} + x_j^2 v_j + x_{j+1}^2 v_{j+1}\right]$$

Hence

$$|Q(Lv)_j| \le Mh^4$$

Therefore

$$|\tau_j(v)| \ \le \ Mh^4 \tag{3.27}$$

Similarly, we obtain

$$|\tau_j(w)| \ \le \ Mh^4 \tag{3.28}$$

For $\tau_j(g)$ we use the form

$$\tau_j(g) = \left(r_j^+ - r_j^-\right) hg'(\xi_1) + \left(q_j^+ b_{j+1} - q_j^- b_{j-1}\right) hg'(\xi_2) \quad : \ x_{j-1} < \xi_i < x_{j+1} \ , \ i = 1, 2$$

Now

$$\left|r_j^+ - r_j^-\right| \le Mh^3/\varepsilon \text{ and } \left|q_j^+ b_{j+1} - q_j^- b_{j-1}\right| \le Mh^3/\varepsilon$$

Therefore using Lemma 3.1, we obtain

$$|\tau_j(g)| \ \le \ Mh^4/\varepsilon \tag{3.29}$$

From (3.27), (3.28) and (3.29), we have

$$|\tau_j(y)| \le Mh^4/\varepsilon \tag{3.30}$$

Estimate of $||A^{-1}||$ :-

Since $r_j^c < 0$ and $r_j^{\pm} > 0$

therefore,

$$||A^{-1}|| \leq \max_j |r_j^- + r_j^c + r_j^+|^{-1} \qquad \text{(Varah [248])}$$

Now for the case $h^2 \leq \varepsilon$, we have

$$\begin{aligned} |r_j^- + r_j^c + r_j^+| &= \left| \left( \frac{p_j}{sinh\ p_j} \right) - (p_j\ coth\ p_j + p_{j+1}\ coth\ p_{j+1}) + \left( \frac{p_{j+1}}{sinh\ p_{j+1}} \right) \right| \\ &= \left| -\frac{p_j^2}{2} - \frac{p_{j+1}^2}{2} \right| + O\left( \frac{h^4}{\varepsilon^2} \right) \\ &\geq M_1 \frac{h^2}{\varepsilon} \end{aligned}$$

$$\Rightarrow \max_j |r_j^- + r_j^c + r_j^+|^{-1} \leq M \frac{\varepsilon}{h^2} \quad \left( \text{where, } M = \frac{1}{M_1} \right)$$

$$\Rightarrow \qquad ||A^{-1}|| \leq M \frac{\varepsilon}{h^2} \tag{3.31}$$

and for the case $h^2 \geq \varepsilon$, we have

$$\begin{aligned} |r_j^- + r_j^c + r_j^+| &\geq M_1 p_j\ coth\ p_j \\ &\geq M_2 p_j^2 \\ &\geq M_3 h^2/\varepsilon \end{aligned}$$

$$\Rightarrow \quad ||A^{-1}|| \leq M \frac{\varepsilon}{h^2} \quad \left( \text{where, } M = \frac{1}{M_3} \right) \tag{3.32}$$

Hence from (3.15), (3.26), (3.30), (3.31) and (3.32), we have the following theorem:

**Theorem 3.1** *Let $b(x)$, $f(x) \in C^2[0,1]$ and $b(x) \geq b > 0, b'(0) = b'(1) = 0$. Let $u_j$, j=0(1)n, be the approximate solution of (3.1), when $a(x) \equiv 0$, obtained using (3.11),(3.12). Then, there is a constant $M$ independent of $\varepsilon$ and $h$ such that*

$$\max_j |y(x_j) - u_j| \leq M h^2$$

## CASE II

For the error analysis of the second and third cases we have used (as in chapter 2) the comparison functions method.

This method uses the following two Lemmas [26]:

**Lemma 3.2** *(maximum principle) : Let* $\{u_j\}$ *be a set of values at the grid points* $x_j$, *satisfying* $u_0 \leq 0, u_n \leq 0$ *and* $Ru_j \geq 0, j = .1(1)n-1$, *then* $u_j \leq 0, j = 0(1)n$.

**Lemma 3.3** *If* $K_1(h,\varepsilon) \geq 0$ *and* $K_2(h,\varepsilon) \geq 0$ *are such that*

$$R(K_1(h,\varepsilon)\phi_j + K_2(h,\varepsilon)\psi_j) \;\geq\; R(\pm e_j) \;=\; \pm\,\tau_j(y)$$

*for each* $j = 1, 2, \cdots, n-1$, *then the discrete maximum principle implies that*

$$|e_j| \;\leq\; K_1(h,\varepsilon)|\phi_j| \;+\; K_2(h,\varepsilon)|\psi_j|$$

*where* $|e_j| = |y(x_j) - u_j|$, *for each* $j$ *and* $\phi$ *and* $\psi$ *are two comparison functions.*

We use the following Lemma [131], for the properties of the exact solution of (3.1) (when $b(x) \equiv 0$):

**Lemma 3.4** *If* $y$ *satisfies (3.1) (when* $b(x) \equiv 0$*), then*

$$y(x) = v(x) + w(x)$$

*where,*

$$v(x) = \left(\frac{\varepsilon y'(1)}{a(1)}\right) exp\left(-\frac{a(1)}{\varepsilon}(1-x)\right)$$

*and*

$$\left|w^{(k)}(x)\right| \leq M\left\{1 + \varepsilon^{-k+1} exp\left(-\frac{c}{\varepsilon}(1-x)\right)\right\}$$

$k = 0(1)4$ *and* $M$ *is a positive constant independent of* $h$ *and* $\varepsilon$.

We use two comparison functions (as in [26]) : $\phi = -2 + x$ and $\psi = -exp(\beta x/\varepsilon)$, ($\beta$ will be taken to be the smallest of various constants appearing in the proof). Therefore $\phi_j = -2 + x_j$ and $\psi_j = -[\mu(\beta)]^j$, $j = 0(1)n$; where, $\mu(\beta) = [r^-(\beta h/\varepsilon)/r^+(\beta h/\varepsilon)] = exp(\beta h/\varepsilon)$.

Remark :- The following inequalities hold:

$$R\phi_j \geq Mh/\varepsilon \ , \ h^2 \leq \varepsilon$$

$$R\phi_j \geq M \ , \ h^2 \geq \varepsilon$$

$$R\psi_j \geq M[\mu(\beta)]^j h/\varepsilon^2 \ , \ h^2 \leq \varepsilon$$

$$R\psi_j \geq M[\mu(\beta)]^j/h \ , \ h^2 \geq \varepsilon$$

Now we estimate the truncation error of the scheme (3.11) using (3.13).

We have,

$$\tau_j(y) = T_0 y_j + T_1 y'_j + T_2 y''_j + \text{Remainder terms}$$

where,

$$\begin{aligned} T_0 &= r_j^- + r_j^c + r_j^+ \\ T_1 &= h(r_j^+ - r_j^-) - (q_j^- a_{j-1} + q_j^c a_j + q_j^+ a_{j+1}) \\ T_2 &= \tfrac{h^2}{2}(r_j^- + r_j^+) + \varepsilon(q_j^- + q_j^c + q_j^+) + h(q_j^- a_{j-1} - q_j^+ a_{j+1}) \end{aligned}$$

Using (3.13) we see that $T_0 = 0, T_1 = 0$ and $|T_2| \leq Mh^2/\varepsilon$

Now from Lemma 3.4, we have $v''_j = -\frac{a(1)}{\varepsilon} y'(1) exp\left(-\frac{a(1)(1-x_j)}{\varepsilon}\right)$, therefore

$$|\tau_j(v)| \leq \frac{Mh^2}{\varepsilon^2} exp\left(-\frac{\delta(1-x_j)}{\varepsilon}\right) \ , \ h^2 \leq \varepsilon$$

Again from Lemma 3.4, we have $|w''_j| \leq M\left[1 + \frac{1}{\varepsilon} exp\left(-\frac{c(1-x_j)}{\varepsilon}\right)\right]$, therefore

$$|\tau_j(w)| \leq M\left[\frac{h^2}{\varepsilon} + \frac{h^2}{\varepsilon^2} exp\left(-\frac{\delta(1-x_j)}{\varepsilon}\right)\right] \ , \ h^2 \leq \varepsilon$$

Now

$$\begin{aligned} \tau_j(y) &= \tau_j(v) + \tau_j(w) \\ \Rightarrow \quad |\tau_j(y)| &\leq M\left[\frac{h^2}{\varepsilon} + \frac{h^2}{\varepsilon^2} exp\left(-\frac{\delta(1-x_j)}{\varepsilon}\right)\right] \ , \ h^2 \leq \varepsilon \end{aligned}$$

The functions $K_1 = h^2$ and $K_2 = h^2/\varepsilon$ satisfies Lemma 3.3 and we obtain

$$\max_j |y_j - u_j| \le Mh^2 \left[1 + \frac{1}{\varepsilon} exp\left(\frac{\delta x_j}{\varepsilon}\right)\right] \quad , \quad h^2 \le \varepsilon$$

In the opposite case, i.e., when $h^2 \ge \varepsilon$, we use the following expression for truncation error :

$$\tau_j(y) = \left(r_j^- + r_j^+\right)\frac{h^2}{2} y''(\xi_1) + \left(q_j^- + q_j^c + q_j^+\right)\varepsilon y''(\xi_2) + \left(q_j^- a_{j-1} - q_j^+ a_{j+1}\right) h y''(\xi_3)$$

$$x_{j-1} < \xi_i < x_{j+1} \ , \ i = 1, 2, 3$$

Now

$$\left|\left(r_j^- + r_j^+\right)\frac{h^2}{2}\right| \le Mh^3 \quad , \quad \left|\left(q_j^- a_{j-1} - q_j^+ a_{j+1}\right) h\right| \le Mh$$

and

$$\left|\left(q_j^- + q_j^c + q_j^+\right)\varepsilon\right| \le Mh^3\left(1 + \frac{h}{\varepsilon^2}\right)$$

Therefore using Lemma 3.4, we obtain

$$|\tau_j(v)| \ \le \ M\frac{h}{\varepsilon^3} exp\left(\frac{-\delta(1 - x_j)}{\varepsilon}\right)$$

and

$$|\tau_j(w)| \ \le \ M\frac{h}{\varepsilon^3}\left[1 + exp\left(\frac{-\delta(1 - x_j)}{\varepsilon}\right)\right]$$

$$\Rightarrow \quad |\tau_j(y)| \ \le \ M\frac{h}{\varepsilon^3}\left[1 + exp\left(\frac{-\delta(1 - x_j)}{\varepsilon}\right)\right]$$

Now the functions $K_1 = h^2$ and $K_2 = h^2 exp(-\delta(1 - x_j)/\varepsilon)$ satisfies Lemma 3.3 and we obtain

$$\max_j |y_j - u_j| \le Mh^2 \left[1 + exp\left(-\frac{\delta(1 - x_j)}{\varepsilon}\right)\right]$$

and since $\beta$ is the smallest positive constant, therefore we obtain the following theorem:

**Theorem 3.2** *Let $\{u_j\}$, $j = 0(1)n$, be a set of values of the approximate solution to $y(x)$ of (3.1), when $b(x) \equiv 0$, obtained using (3.11) and (3.13). Then there are positive constants $\delta$ ($> \beta$) and $M$ (independent of $h$ and $\varepsilon$) such that the following estimate holds :*

$$\max_j |y(x_j) - u_j| \ \le \ \begin{cases} Mh^2 \left[1 + \frac{1}{\varepsilon} exp\left(\frac{\delta x_j}{\varepsilon}\right)\right] & , \ h^2 \le \varepsilon \\ Mh^2 \left[1 + exp\left(-\frac{\delta(1 - x_j)}{\varepsilon}\right)\right] & , \ h^2 \ge \varepsilon \end{cases}$$

### CASE III

We use the same approach as was in CASE II and the same Lemma, i.e., Lemma 3.4, for the properties of the exact solution of (3.1) (as in this case also the boundary layer will be at the right end of the underlying interval).

In this case we observe that :

$$R\phi_j \geq Mh^2/\varepsilon \ , \ h^2 \leq \varepsilon$$

$$R\phi_j \geq Mh \ , \ h^2 \geq \varepsilon$$

$$R\psi_j \geq M[\mu(\beta)]^j h/\varepsilon \ , \ h^2 \leq \varepsilon$$

$$R\psi_j \geq M[\mu(\beta)]^j h \ , \ h^2 \geq \varepsilon$$

and

$$\Rightarrow \quad |\tau_j(y)| \leq \begin{cases} M\frac{h^2}{\varepsilon}\left[1 + exp\left(-\frac{a(1)}{\varepsilon}(1-x_j)\right)\right] & , \ h^2 \leq \varepsilon \\ M\frac{h^2}{\varepsilon^2}\left[1 + exp\left(-\frac{a(1)}{\varepsilon}(1-x_j)\right)\right] & , \ h^2 \geq \varepsilon \end{cases}$$

In the case $h^2 \leq \varepsilon$, the functions $K_1 = h^2$ and $K_2 = h^2 exp(-a(1)/\varepsilon)$ satisfies Lemma 3.3 and we obtain

$$\max_j |y_j - u_j| \leq Mh^2\left[1 + exp\left(\frac{\beta x_j - a(1)}{\varepsilon}\right)\right]$$

and in the case $h^2 \geq \varepsilon$, the functions $K_1 = h^2$ and $K_2 = h^2$ satisfies Lemma 3.3 and we obtain

$$\max_j |y_j - u_j| \leq Mh^2\left[1 + exp\left(\frac{\beta x_j}{\varepsilon}\right)\right]$$

and since $\beta$ is the smallest positive constant, therefore we obtain the following theorem:

**Theorem 3.3** *Let $\{u_j\}$, $j = 0(1)n$, be a set of values of the approximate solution to $y(x)$ of (3.1) obtained using (3.11) and (3.14). Then there are positive constants $\delta$ $(> \beta)$ and $M$ (independent of $h$ and $\varepsilon$) such that the following estimate holds :*

$$\max_j |y(x_j) - u_j| \ \leq \begin{cases} Mh^2\left[1 + exp\left(\frac{\delta x_j - a(1)}{\varepsilon}\right)\right] & , \ h^2 \leq \varepsilon \\ Mh^2\left[1 + exp\left(\frac{\delta x_j}{\varepsilon}\right)\right] & , \ h^2 \geq \varepsilon \end{cases}$$

## 3.5 Test Examples and Numerical Results

In this section we present some numerical results which illustrate the predicted theory.

**Example 3.1** [66] :

$$-\varepsilon y'' + y = -(cos^2\pi x + 2\varepsilon\pi^2 cos2\pi x) \; ; \; y(0) = 0 \;\; , \;\; y(1) = 0$$

*Its exact solution is given by*

$$y(x) = \frac{exp[-(1-x)/\sqrt{\varepsilon}\,] + exp[-x/\sqrt{\varepsilon}\,]}{1 + exp[-1/\sqrt{\varepsilon}\,]} - cos^2\pi x$$

**Example 3.2** [253] :

$$-\varepsilon y'' + (1 + x(1-x))y = f(x) \; ; \; y(0) = 0 \;\; , \;\; y(1) = 0$$

*where,*

$$\begin{aligned} f(x) &= 1 + x(1-x) + \left[2\sqrt{\varepsilon} \; - \; x^2(1-x)\right] exp[-(1-x)/\sqrt{\varepsilon}\,] \\ &\quad + \left[2\sqrt{\varepsilon} \; - \; x(1-x)^2\right] exp[-x/\sqrt{\varepsilon}\,] \end{aligned}$$

*Its exact solution is given by*

$$y(x) \;=\; 1 + (x-1)\; exp[-x/\sqrt{\varepsilon}\,] - x\; exp[-(1-x)/\sqrt{\varepsilon}\,]$$

**Example 3.3** [153] :

$$-\varepsilon y'' + y' = 0 \; ; \; y(0) = 1 \;\; , \;\; y(1) = 0$$

*Its exact solution is given by*

$$y(x) \;=\; \frac{1 - exp[(x-1)/\varepsilon]}{1 - exp(-1/\varepsilon)}$$

**Example 3.4** [153] :

$$-\varepsilon y'' + y' = exp(x) \; ; \; y(0) = 0 \;\; , \;\; y(1) = 0$$

*Its exact solution is given by*

$$y(x) \;=\; \frac{1}{1-\varepsilon}\left[exp(x) - \frac{1 - exp\{1 - (1/\varepsilon)\} + \{exp(1) - 1\}exp\{(x-1)/\varepsilon\}}{1 - exp(-1/\varepsilon)}\right]$$

**Example 3.5** *[55]* :

$$-\varepsilon y'' + y' + (1+\varepsilon)y = 0 \; ; \; y(0) = 1 + exp[-(1+\varepsilon)/\varepsilon] \;\; , \;\; y(1) = 1 + [1/exp(1)]$$

*Its exact solution is given by*

$$y(x) \;\; = \;\; exp[(1+\varepsilon)(x-1)/\varepsilon] + exp(-x)$$

**Example 3.6** *[74]* :

$$-\varepsilon y'' + \frac{1}{x+1}y' + \frac{1}{x+2}y = f(x) \; ; \; y(0) = 1 + 2^{-1/\varepsilon} \;\; , \;\; y(1) = exp(1) + 2$$

*where,*

$$f(x) \;\; = \;\; \left(-\varepsilon + \frac{1}{x+1} + \frac{1}{x+2}\right) exp(x) + \frac{1}{x+2}2^{-1/\varepsilon}(x+1)^{(1+1/\varepsilon)}$$

*Its exact solution is given by*

$$y(x) \;\; = \;\; exp(x) + 2^{-1/\varepsilon}(x+1)^{(1+1/\varepsilon)}$$

Tables 3.1, 3.2, 3.4, 3.5, 3.7 and 3.8 contain the maximum errors at all the mesh points :

$$\max_j |y(x_j) - u_j| \tag{3.33}$$

for different $n$ and $\varepsilon$, $n = 1/h$ .

Tables 3.3, 3.6 and 3.9 contain the numerical rate of uniform convergence which is determined as in [66] :

$$r_{k,\varepsilon} = log_2\left(\frac{z_{k,\varepsilon}}{z_{k+1,\varepsilon}}\right) \;\; , \;\; k = 0, 1, 2, \cdots\cdots\cdots$$

where,

$$z_{k,\varepsilon} = \max_j |u_j^{h/2^k} \;\; - \;\; u_{2j}^{h/2^{k+1}}| \;\; , \;\; k = 0, 1, 2, \cdots\cdots\cdots$$

and $u_j^{h/2^k}$ denotes the value of $u_j$ for the mesh length $h/2^k$.

Table 3.1: Max. Errors for Example 3.1

| $\varepsilon$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ | $n=512$ | $n=1024$ |
|---|---|---|---|---|---|---|---|
| $1.0E-1$ | 0.17E-14 | 0.44E-14 | 0.36E-14 | 0.76E-13 | 0.13E-13 | 0.21E-11 | 0.12E-10 |
| $1.0E-2$ | 0.44E-15 | 0.10E-14 | 0.77E-14 | 0.18E-13 | 0.41E-13 | 0.39E-12 | 0.22E-11 |
| $1.0E-3$ | 0.33E-15 | 0.33E-15 | 0.22E-14 | 0.17E-14 | 0.14E-13 | 0.12E-12 | 0.26E-12 |
| $1.0E-4$ | 0.11E-15 | 0.11E-15 | 0.22E-15 | 0.33E-15 | 0.30E-14 | 0.40E-14 | 0.32E-13 |
| $1.0E-5$ | 0.00E+00 | 0.33E-15 | 0.11E-15 | 0.22E-15 | 0.22E-15 | 0.16E-14 | 0.11E-14 |
| $1.0E-6$ | 0.00E+00 | 0.11E-15 | 0.11E-15 | 0.11E-15 | 0.22E-15 | 0.44E-15 | 0.33E-15 |
| $1.0E-7$ | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.11E-15 | 0.11E-15 | 0.22E-15 |
| $1.0E-8$ | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.11E-15 | 0.22E-15 |
| $1.0E-9$ | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 |

Table 3.2: Max. Errors for Example 3.2

| $\varepsilon$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ | $n=512$ | $n=1024$ |
|---|---|---|---|---|---|---|---|
| $1.0E-1$ | 0.10E-02 | 0.25E-03 | 0.63E-04 | 0.16E-04 | 0.39E-05 | 0.98E-06 | 0.24E-06 |
| $1.0E-2$ | 0.32E-02 | 0.79E-03 | 0.20E-03 | 0.50E-04 | 0.12E-04 | 0.31E-05 | 0.78E-06 |
| $1.0E-3$ | 0.86E-02 | 0.29E-02 | 0.69E-03 | 0.17E-03 | 0.43E-04 | 0.11E-04 | 0.27E-05 |
| $1.0E-4$ | 0.48E-02 | 0.42E-02 | 0.21E-02 | 0.58E-03 | 0.14E-03 | 0.35E-04 | 0.87E-05 |
| $1.0E-5$ | 0.15E-02 | 0.15E-02 | 0.16E-02 | 0.12E-02 | 0.46E-03 | 0.11E-03 | 0.28E-04 |
| $1.0E-6$ | 0.48E-03 | 0.49E-03 | 0.49E-03 | 0.50E-03 | 0.48E-03 | 0.29E-03 | 0.94E-04 |
| $1.0E-7$ | 0.15E-03 | 0.15E-03 | 0.16E-03 | 0.16E-03 | 0.16E-03 | 0.16E-03 | 0.14E-03 |
| $1.0E-8$ | 0.48E-04 | 0.49E-04 | 0.49E-04 | 0.50E-04 | 0.50E-04 | 0.50E-04 | 0.50E-04 |
| $1.0E-9$ | 0.15E-04 | 0.15E-04 | 0.16E-04 | 0.16E-04 | 0.16E-04 | 0.16E-04 | 0.16E-04 |

Table 3.3: Rate of Convergence for Example 3.2
$n = 32, 64, 128, 256, 512$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/2 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/4 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/8 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/16 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/32 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/64 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/128 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/256 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |

Table 3.4: Max. Errors for Example 3.3

| $\varepsilon$ | $n=8$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ | $n=512$ |
|---|---|---|---|---|---|---|---|
| 1/4 | 0.33E-15 | 0.32E-14 | 0.83E-14 | 0.13E-13 | 0.35E-13 | 0.73E-12 | 0.22E-11 |
| 1/8 | 0.11E-15 | 0.17E-14 | 0.80E-14 | 0.31E-13 | 0.10E-13 | 0.45E-13 | 0.23E-11 |
| 1/16 | 0.33E-15 | 0.11E-15 | 0.50E-14 | 0.19E-13 | 0.86E-13 | 0.15E-13 | 0.59E-13 |
| 1/32 | 0.11E-15 | 0.12E-14 | 0.33E-15 | 0.12E-13 | 0.40E-13 | 0.18E-12 | 0.32E-13 |
| 1/64 | 0.11E-15 | 0.11E-15 | 0.41E-14 | 0.29E-14 | 0.43E-13 | 0.48E-13 | 0.78E-12 |
| 1/128 | 0.00E+00 | 0.11E-15 | 0.11E-15 | 0.11E-13 | 0.98E-14 | 0.11E-12 | 0.48E-13 |
| 1/256 | 0.00E+00 | 0.00E+00 | 0.11E-15 | 0.11E-15 | 0.25E-13 | 0.24E-13 | 0.25E-12 |
| 1/512 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.11E-15 | 0.11E-15 | 0.54E-13 | 0.52E-13 |
| 1/1000 | 0.00E+00 | 0.00E+00 | 0.11E-15 | 0.11E-15 | 0.22E-15 | 0.28E-13 | 0.78E-15 |

Table 3.5: Max. Errors for Example 3.4

| $\varepsilon$ | $n=8$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ | $n=512$ |
|---|---|---|---|---|---|---|---|
| 1/4 | 0.18E-02 | 0.45E-03 | 0.11E-03 | 0.28E-04 | 0.70E-05 | 0.18E-05 | 0.44E-06 |
| 1/8 | 0.26E-02 | 0.66E-03 | 0.16E-03 | 0.41E-04 | 0.10E-04 | 0.26E-05 | 0.64E-06 |
| 1/16 | 0.32E-02 | 0.81E-03 | 0.21E-03 | 0.51E-04 | 0.13E-04 | 0.32E-05 | 0.80E-06 |
| 1/32 | 0.34E-02 | 0.89E-03 | 0.23E-03 | 0.58E-04 | 0.15E-04 | 0.36E-05 | 0.91E-06 |
| 1/64 | 0.29E-02 | 0.91E-03 | 0.24E-03 | 0.62E-04 | 0.16E-04 | 0.39E-05 | 0.98E-06 |
| 1/128 | 0.25E-02 | 0.80E-03 | 0.24E-03 | 0.64E-04 | 0.16E-04 | 0.41E-05 | 0.10E-05 |
| 1/256 | 0.22E-02 | 0.68E-03 | 0.21E-03 | 0.61E-04 | 0.16E-04 | 0.42E-05 | 0.11E-05 |
| 1/512 | 0.20E-02 | 0.60E-03 | 0.18E-03 | 0.53E-04 | 0.15E-04 | 0.42E-05 | 0.11E-05 |
| 1/1000 | 0.19E-02 | 0.55E-03 | 0.16E-03 | 0.46E-04 | 0.14E-04 | 0.39E-05 | 0.10E-05 |

Table 3.6: Rate of Convergence for Example 3.4
$n = 32, 64, 128, 256, 512$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/2 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/4 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/8 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/16 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/32 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/64 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/128 | 0.19E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/256 | 0.17E+01 | 0.19E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.19E+01 |

Table 3.7: Max. Errors for Example 3.5

| $\varepsilon$ | $n=8$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ | $n=512$ |
|---|---|---|---|---|---|---|---|
| 1/4 | 0.11E-15 | 0.89E-15 | 0.15E-13 | 0.68E-13 | 0.13E-12 | 0.25E-12 | 0.14E-11 |
| 1/8 | 0.67E-15 | 0.12E-14 | 0.18E-14 | 0.22E-13 | 0.14E-12 | 0.94E-12 | 0.34E-11 |
| 1/16 | 0.44E-15 | 0.67E-15 | 0.46E-14 | 0.47E-14 | 0.65E-13 | 0.29E-12 | 0.19E-11 |
| 1/32 | 0.61E-15 | 0.14E-14 | 0.35E-14 | 0.74E-14 | 0.11E-14 | 0.89E-14 | 0.61E-12 |
| 1/64 | 0.78E-15 | 0.11E-14 | 0.78E-15 | 0.78E-14 | 0.27E-13 | 0.57E-14 | 0.20E-12 |
| 1/128 | 0.94E-15 | 0.67E-15 | 0.39E-15 | 0.21E-14 | 0.10E-14 | 0.34E-13 | 0.12E-12 |
| 1/256 | 0.22E-15 | 0.22E-15 | 0.46E-14 | 0.42E-14 | 0.16E-13 | 0.23E-14 | 0.19E-12 |
| 1/512 | 0.22E-15 | 0.16E-14 | 0.67E-15 | 0.42E-14 | 0.44E-15 | 0.33E-13 | 0.11E-13 |
| 1/1000 | 0.11E-15 | 0.11E-15 | 0.67E-15 | 0.11E-14 | 0.44E-15 | 0.13E-13 | 0.31E-13 |

Table 3.8: Max. Errors for Example 3.6

| $\varepsilon$ | $n=8$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ | $n=512$ |
|---|---|---|---|---|---|---|---|
| 1/4 | 0.48E-03 | 0.12E-03 | 0.30E-04 | 0.75E-05 | 0.19E-05 | 0.47E-06 | 0.12E-06 |
| 1/8 | 0.20E-02 | 0.52E-03 | 0.13E-03 | 0.33E-04 | 0.82E-05 | 0.20E-05 | 0.51E-06 |
| 1/16 | 0.75E-02 | 0.18E-02 | 0.44E-03 | 0.11E-03 | 0.28E-04 | 0.70E-05 | 0.17E-05 |
| 1/32 | 0.14E-01 | 0.45E-02 | 0.11E-02 | 0.27E-03 | 0.67E-04 | 0.17E-04 | 0.42E-05 |
| 1/64 | 0.14E-01 | 0.82E-02 | 0.25E-02 | 0.60E-03 | 0.15E-03 | 0.37E-04 | 0.92E-05 |
| 1/128 | 0.75E-02 | 0.80E-02 | 0.43E-02 | 0.13E-02 | 0.31E-03 | 0.77E-04 | 0.19E-04 |
| 1/256 | 0.25E-02 | 0.45E-02 | 0.42E-02 | 0.22E-02 | 0.66E-03 | 0.16E-03 | 0.39E-04 |
| 1/512 | 0.22E-02 | 0.19E-02 | 0.24E-02 | 0.22E-02 | 0.11E-02 | 0.33E-03 | 0.80E-04 |
| 1/1000 | 0.24E-02 | 0.62E-03 | 0.12E-02 | 0.13E-02 | 0.11E-02 | 0.56E-03 | 0.16E-03 |

Table 3.9: Rate of Convergence for Example 3.6
$n = 32, 64, 128, 256, 512$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/1 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/2 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/4 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/8 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/16 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/32 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/64 | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/128 | 0.18E+01 | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/256 | 0.11E+01 | 0.18E+01 | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.18E+01 |

## 3.6 Discussion

We have described a numerical method for solving singular perturbation problems using spline in tension. It is a practical method and can easily be implemented on a computer to solve such problems. The method has been analysed for convergence. Test examples have been solved to demonstrate the efficiency of the proposed method.

Table 3.3 shows the rate of uniform convergence as 2 for the problems of the type as considered in the first case, whereas Tables 3.6 and 3.9 shows the rates as 1.9 and 1.8 for the problems of the type which are considered in the cases second and third respectively.

Example 3.6 has been solved earlier by Eugene [74] using finite difference techniques. Our results are comparable with those obtained by Eugene with his $O(h^p)$, with $p = 2$, exponentially fitted $HODIE$ schemes. However, we obtained better results than those of his $O(h)$ exponentially fitted $HODIE$ schemes.

# Chapter 4

## VARIABLE MESH AND EXPONENTIALLY FITTED SPLINE IN COMPRESSION FOR SINGULAR PERTURBATION PROBLEMS

## 4.1 Introduction

Consider the singularly perturbed problem

$$\left.\begin{array}{l} Ly \equiv \varepsilon y'' + a(x)y' + b(x)y = f(x) \\ y(0) = \alpha_0, \;\; y(1) = \alpha_1, \;\; \alpha_0, \alpha_1 \in R \end{array}\right\} \tag{4.1}$$

where, $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth with $a(x) > c > 0$, $b(x) > 0$, $c$ is some constant and $\varepsilon$ is a small positive parameter.

It is well known that the classical cubic spline collocation method when applied to (4.1) (with $b(x) \equiv 0$), have an inherent formal cell Reynolds number limitation, i.e., $ha(x_j)/2\varepsilon$ must be less or equal to 1. For small $\varepsilon$ this leads to spurious oscillations or gross inaccuracies in the approximate solution. In order to avoid these difficulties one introduces exponential functions into the spline basis, e.g., Flaherty and Mathon [80] used polynomial and tension splines, Chin and Krasny [51] used $\gamma$ -elliptic splines, Jain and Aziz [111] used adaptive splines. Stynes and O'Riordan [230] used finite element techniques to tackle such problems.

There are two possibilities to obtain small truncation error inside the boundary layer. The first is to choose a fine mesh there whereas the second one is to choose a difference formula reflecting the behaviour of the solution inside the boundary layer. Thus in this chapter we gave two methods, namely, VMSC (Variable Mesh Spline in Compression) and EFSC (Exponentially Fitted Spline in Compression) which are based on the above idea. By making use of the continuity of the first order derivative of the spline function,

the resulting spline difference schemes gives a tridiagonal system which can be solved efficiently by the well known algorithms.

In Section 4.2 we give a brief description of these methods. The derivation of the difference schemes has been given in Section 4.3 along with the mesh selection strategy and the determination of the fitting factor. In Section 4.4, the second order accuracy of the method is shown. To demonstrate the applicability of the proposed method several numerical examples have been solved in Section 4.5 and the results are presented along with their comparison with those obtained without using variable mesh. Finally, the discussion is given in Section 4.6.

## 4.2 Description of the Methods

Description of the Method for VMSC:

For $x \in [x_{j-1}, x_j]$, we define $\widetilde{a}(x) = (a_{j-1} + a_j)/2$ and anologously $\widetilde{f}(x)$ too. Following the ideas of spline in compression from Jain [110], the approximate solution of the problem (4.1) (when $b(x) \equiv 0$), is sought in the form of the function S(x), which on each interval $[x_{j-1}, x_j]$, (denoted by $S_j(x)$) satisfies the following relations :-

(i) the differential equation

$$\varepsilon S_j''(x) + \widetilde{a}(x) S_j(x) = \widetilde{f}(x) \tag{4.2}$$

(ii) the interpolating conditions

$$S_j(x_{j-1}) = u_{j-1}, \quad S_j(x_j) = u_j \tag{4.3}$$

(iii) the continuity condition

$$S_j'(x_j^+) = S_j'(x_j^-) \tag{4.4}$$

(iv) the consistency condition

$$\frac{p_j}{2} = tan\frac{p_j}{2}, \quad p_j = h_j \left( \frac{a_{j-1} + a_j}{2\varepsilon} \right) \tag{4.5}$$

where,

$$x \in [x_{j-1}, x_j], \quad h_j = x_j - x_{j-1}, \quad j = 1(1)n$$

Solving equation (4.2) with the help of (4.3), we obtain

$$\begin{aligned} S_j(x) &= \frac{1}{F_j}\left[D_j\, exp\left(-\frac{\alpha_j x_{j-1}}{\varepsilon}\right) - E_j\, exp\left(-\frac{\alpha_j x_j}{\varepsilon}\right)\right] + \frac{E_j - D_j}{F_j}\, exp\left(-\frac{\alpha_j x}{\varepsilon}\right) \\ &+\frac{\gamma_j x}{\alpha_j} - \frac{\gamma_j \varepsilon}{\alpha_j^2} \end{aligned} \tag{4.6}$$

where,

$$F_j = \left[exp\left(-\frac{\alpha_j x_{j-1}}{\varepsilon}\right) - exp\left(-\frac{\alpha_j x_j}{\sigma_j}\right)\right]$$

$$D_j = u_j - \frac{\gamma_j x_j}{\alpha_j} + \frac{\gamma_j \varepsilon}{\alpha_j^2}\ ,\ E_j = u_{j-1} - \frac{\gamma_j x_{j-1}}{\alpha_j} + \frac{\gamma_j \varepsilon}{\alpha_j^2}$$

$$\alpha_j = \frac{a_{j-1} + a_j}{2}\quad ,\quad \gamma_j = \frac{f_{j-1} + f_j}{2}$$

Using this spline function we will derive the difference scheme in Section 4.3.

Description of the Method for EFSC:

For $x \in [x_{j-1}, x_j]$, we define $\widetilde{a}(x) = (a_{j-1} + a_j)/2$ and anologously $\widetilde{b}(x)$ and $\widetilde{f}(x)$ too.

Consider first the equation (4.1), with $a(x) \equiv 0$. Therefore we need to solve

$$\left.\begin{array}{l} Ly \;\equiv\; \varepsilon y'' \;+\; b(x)y \;=\; f(x) \\ y(0) \;=\; \alpha_0,\;\; y(1) \;=\; \alpha_1,\;\; \alpha_0, \alpha_1 \in R \end{array}\right\} \tag{4.7}$$

We define the fitting comparison problem associated with (4.7) by:

$$\left.\begin{array}{l} Ly \;\equiv\; \sigma(x,\varepsilon) y'' \;+\; b(x)y \;=\; f(x) \\ y(0) \;=\; \alpha_0,\;\; y(1) \;=\; \alpha_1,\;\; \alpha_0, \alpha_1 \in R \end{array}\right\} \tag{4.8}$$

where, $\sigma(x, \varepsilon)$ is an exponential fitting factor which is to be determined subsequently.

The approximate solution of this problem, is sought in the form of the function S(x), which on each interval $[x_{j-1}, x_j]$, (denoted by $S_j(x)$) satisfies the following relations :-

(i) the differential equation

$$\sigma_j S_j''(x) \;+\; \widetilde{b}(x) S_j(x) \;=\; \widetilde{f}(x) \tag{4.9}$$

(ii) the interpolating conditions

$$S_j(x_{j-1}) \;=\; u_{j-1},\;\; S_j(x_j) \;=\; u_j \tag{4.10}$$

(iii) the continuity condition

$$S'_j(x_j^+) \;=\; S'_j(x_j^-) \tag{4.11}$$

(iv) the consistency condition

$$\frac{p_j}{2} \;=\; tan\frac{p_j}{2}, \quad p_j \;=\; h\sqrt{\frac{b_{j-1}+b_j}{2\sigma_j}} \tag{4.12}$$

where,

$$x \in [x_{j-1}, x_j], \;\; x_j \;=\; jh, \;\; j \;=\; 1(1)n, \;\; h \;=\; 1/n.$$

Solving equation (4.9) with the help of (4.10), we obtain

$$S_j(x) = \frac{1}{-sin\ g_j h}\left[A_j\ sin\ g_j(x_{j-1}-x) + B_j\ sin\ g_j(x-x_j)\right] + \frac{\gamma_j}{\beta_j} \tag{4.13}$$

where,

$$A_j = u_j - \tfrac{\gamma_j}{\beta_j}, \;\; B_j = u_{j-1} - \tfrac{\gamma_j}{\beta_j}, \;\; g_j = \sqrt{\tfrac{\beta_j}{\sigma_j}}, \;\; \beta_j = \tfrac{b_{j-1}+b_j}{2}, \;\; \gamma_j = \tfrac{f_{j-1}+f_j}{2}.$$

Equation (4.13) together with (4.12) is known as spline in compression ([110]). Using this spline function we will derive the difference scheme in Section 4.3.

For the second case, we need to solve (4.1) (when $b(x) \equiv 0$), i.e.,

$$\left.\begin{array}{l} Ly \;\equiv\; \varepsilon y'' \;+\; a(x)y' \;=\; f(x) \\ y(0) \;=\; \alpha_0, \;\; y(1) \;=\; \alpha_1, \;\; \alpha_0, \alpha_1 \in R \end{array}\right\} \tag{4.14}$$

The approximate solution of the problem (4.14), we seek (as in the first case) as a solution of the differential equation

$$\sigma_j S''_j(x) \;+\; \widetilde{a}(x)S'_j(x) \;=\; \widetilde{f}(x) \tag{4.15}$$

whereas, in this case the parameter $p_j$ used in (4.12) will be given by:

$$p_j = h(a_{j-1}+a_j)/2\sigma_j$$

Solving (4.15) with the help of (4.10), we obtain

$$\begin{aligned} S_j(x) \;=\;& \frac{1}{F_j}\left[D_j\ exp\left(-\frac{\alpha_j x_{j-1}}{\sigma_j}\right) - E_j\ exp\left(-\frac{\alpha_j x_j}{\sigma_j}\right)\right] + \frac{E_j - D_j}{F_j}\ exp\left(-\frac{\alpha_j x}{\sigma_j}\right) \\ & + \frac{\gamma_j x}{\alpha_j} - \frac{\gamma_j \sigma_j}{\alpha_j^2} \end{aligned} \tag{4.16}$$

where,

$$F_j = \left[ exp\left(-\frac{\alpha_j x_{j-1}}{\sigma_j}\right) - exp\left(-\frac{\alpha_j x_j}{\sigma_j}\right)\right]$$

$$D_j = u_j - \frac{\gamma_j x_j}{\alpha_j} + \frac{\gamma_j \sigma_j}{\alpha_j^2} \ , \ E_j = u_{j-1} - \frac{\gamma_j x_{j-1}}{\alpha_j} + \frac{\gamma_j \sigma_j}{\alpha_j^2} \ , \ \alpha_j = \frac{a_{j-1} + a_j}{2}$$

## 4.3 Derivation of the schemes

Derivation of the scheme for VMSC:

Since $S(x) \in C^2[0,1]$, therefore we have

$$S'_j(x_j) \ = \ S'_{j+1}(x_j) \tag{4.17}$$

Differentiating (4.6) with respect to $x$, putting $x = x_j$ and using (4.17), we obtain the difference scheme

$$Ru_j \ = \ Qf_j \ , \ j = 1(1)n-1 \tag{4.18}$$

where,

$$Ru_j \ = \ r_j^- u_{j-1} \ + \ r_j^c u_j \ + \ r_j^+ u_{j+1}$$

$$Qf_j \ = \ q_j^- f_{j-1} \ + \ q_j^c f_j \ + \ q_j^+ f_{j+1}$$

$$u_0 \ = \ \alpha_0 \ , \ u_n \ = \ \alpha_1$$

$$r_j^- = \frac{1}{h_j}\left(1 - \frac{p_j}{2}\right) \ , \ r_j^+ = \frac{1}{h_{j+1}}\left(1 + \frac{p_{j+1}}{2}\right) \ , \ r_j^c = -(r_j^- + r_j^+)$$

$$q_j^- = \frac{h_j}{4\varepsilon} \ , \ q_j^+ = \frac{h_{j+1}}{4\varepsilon} \ , \ q_j^c = q_j^- \ + \ q_j^+ \tag{4.19}$$

Mesh Selection Strategy:

We form the non-uniform grid in such a way that more points are generated in the boundary layer region than outside to it.

On the interval $[0,\delta]$, where $\delta$ denotes the boundary layer width, the grid is non-uniform and is defined as follows:

Let $n_1 < n$ be the number of mesh points in the boundary layer region $[0,\delta]$ and let the positive constants $\widetilde{h}_1$ and $K$ be known. Then we generate the mesh as :

$\widetilde{h}_j = \widetilde{h}_{j-1} + K(\widetilde{h}_{j-1}/\varepsilon)\min(\widetilde{h}_{j-1}^2, \varepsilon)$ , $j = 2(1)n_1$.

Now, let

$\widetilde{q} = \sum_{j=1}^{n_1} \widetilde{h}_j$

$q = \frac{\delta}{\widetilde{q}}$

and define

$x_0 = 0$

$h_j = q\widetilde{h}_j$ , $j = 1(1)n_1$

$x_j = x_{j-1} + h_j$ , $j = 1(1)n_1$

On the interval $[\delta, 1]$, the grid is uniform and is defined as :

$h_j = \frac{1-\delta}{n_2}$ , $j = n_1 + 1, n$ , where $n_2 = n - n_1$

$x_j = x_{j-1} + h_j$ , $j = n_1 + 1, n$ .

<u>Derivation of the scheme for EFSC</u>:
Since $S(x) \in C^2[0,1]$, therefore we have

$$S_j'(x_j) = S_{j+1}'(x_j) \tag{4.20}$$

**CASE I**

Differentiating (4.13) with respect to $x$ and putting $x = x_j$ and using (4.20), we obtain the difference scheme

$$Ru_j = Qf_j \ , \ j = 1(1)n-1 \tag{4.21}$$

where,

$$Ru_j = r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1}$$

$$Qf_j = q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1}$$

$$u_0 = \alpha_0 , \; u_n = \alpha_1$$

$$r_j^- = 1 + \frac{p_j^2}{4},\; r_j^+ = 1 + \frac{p_{j+1}^2}{4},\; r_j^c = -4 + r_j^- + r_j^+,\; q_j^- = \frac{h^2}{4\sigma_j},\; q_j^+ = \frac{h^2}{4\sigma_{j+1}},\; q_j^c = q_j^- + q_j^+$$

$$p_j = h\sqrt{\frac{\beta_j}{\sigma_j}} , \; \beta_j = \frac{b_{j-1} + b_j}{2} \tag{4.22}$$

where, $\sigma_j$ is to be determined.

**CASE II**

Differentiating (4.16) with respect to $x$ and putting $x = x_j$ and using (4.20), we obtain the difference scheme given by (4.21), where

$$r_j^- = 1 - \frac{p_j}{2},\; r_j^+ = 1 + \frac{p_{j+1}}{2},\; r_j^c = -(r_j^- + r_j^+),\; q_j^- = \frac{h^2}{4\sigma_j},\; q_j^+ = \frac{h^2}{4\sigma_{j+1}},\; q_j^c = q_j^- + q_j^+$$

$$p_j = h\frac{\alpha_j}{\sigma_j} , \; \alpha_j = \frac{a_{j-1} + a_j}{2} \tag{4.23}$$

where, $\sigma_j$ is to be determined.

Remark:- If we do not use fitting factor then instead of (4.22), we obtain

$$r_j^- = 1 + \frac{p_j^2}{4},\; r_j^+ = 1 + \frac{p_{j+1}^2}{4},\; r_j^c = -4 + r_j^- + r_j^+,\; q_j^- = q_j^+ = \frac{h^2}{4\varepsilon},\; q_j^c = \frac{h^2}{2\varepsilon}$$

$$p_j = h\sqrt{\frac{\beta_j}{\varepsilon}} , \; \beta_j = \frac{b_{j-1} + b_j}{2} \tag{4.24}$$

and similarly instead of (4.23), we obtain

$$r_j^- = 1 - \frac{p_j}{2},\; r_j^+ = 1 + \frac{p_{j+1}}{2},\; r_j^c = -(r_j^- + r_j^+),\; q_j^- = q_j^+ = \frac{h^2}{4\varepsilon},\; q_j^c = \frac{h^2}{2\varepsilon}$$

$$p_j = h\frac{\alpha_j}{\varepsilon} , \; \alpha_j = \frac{a_{j-1} + a_j}{2} \tag{4.25}$$

Determination of the Fitting factor

**CASE I**

To obtain a suitable fitting factor $\sigma(x, \varepsilon)$ for this case, we shall use the following Lemma :-

**Lemma 4.1** *[66] Let $y(x) \in C^4[0,1]$. Let $b'(0) = b'(1) = 0$. Then the solution of the problem (4.1) (with $a(x) \equiv 0$) has the form :-*

$$y(x) = v(x) + w(x) + g(x)$$

*where,*

$$v(x) = q_0 exp\left[-x\{b(0)/(-\varepsilon)\}^{\frac{1}{2}}\right] \quad , \quad w(x) = q_1 exp\left[-(1-x)\{b(1)/(-\varepsilon)\}^{\frac{1}{2}}\right]$$

*$q_0$ and $q_1$ are bounded functions of $\varepsilon$ independent of $x$ and*

$$|g^{(k)}(x)| \leq N\left(1 + (-\varepsilon)^{1-\frac{k}{2}}\right) \quad , \quad k = 0(1)4$$

*$N$ is a constant independent of $\varepsilon$.*

We require that the truncation error for the boundary layer functions should be equal to zero when $b(x) = b =$ constant.

We take a fitting factor in the following way:

$$\sigma_j = \frac{h^2\beta_j}{4}\,\mu(\rho)$$

where $\mu(\rho)$, $(\rho = \sqrt{b/\varepsilon})$ is to be determined.

From the condition $Rv_j = 0$ for $b(x) = b =$ constant, we have

$$\mu(\rho) = cot^2\left(\frac{\rho h}{2}\right)$$

The condition $Rw_j = 0$ for $b(x) = b =$ constant, will give the same $\mu(\rho)$. Therefore we define

$$\mu(\rho) = cot^2\left(\frac{\rho h}{2}\right) \quad , \text{ when } b(x) = b = \text{constant}$$

and

$$\mu(\rho_j) = cot^2\left(\frac{\rho_j h}{2}\right) \quad , \text{ when } b(x) \neq \text{constant}$$

Hence the variable fitting factor $\sigma_j$ is defined as:

$$\sigma_j = \frac{h^2\beta_j}{4}\,\mu(\rho_j) \tag{4.26}$$

**CASE II**

In this case instead of Lemma 4.1, we use the following Lemma to obtain a suitable fitting factor:-

**Lemma 4.2** *[26] Let $a(x), f(x) \in C^3(0,1)$, then*

$$y(x) \;=\; v(x) \;+\; w(x)$$

*where,* $v(x) \;=\; \left(-\varepsilon\frac{y'(0)}{a(0)}\right) exp\left(-\frac{a(0)x}{\varepsilon}\right)$ *and* $|w^{(k)}(x)| \;\leq\; M\left[1+\varepsilon^{-k+1}exp(-\frac{\delta x}{\varepsilon})\right]$ , $k = 0(1)4, \delta = a/4$, *where* $0 < a < a(x)$ *for all* $x$ *and* $M$ *is a positive constant independent of* $h$ *and* $\varepsilon$.

We require that the truncation error for the boundary layer function should be equal to zero when $a(x) = a =$ constant.

We take a fitting factor in the following way:

$$\sigma_j = \frac{h\alpha_j}{2}\,\mu(\rho)$$

where $\mu(\rho)$, $(\rho = a/\varepsilon)$ is to be determined.

From the condition $Rv_j = 0$ for $a(x) = a =$ constant, we have

$$\mu(\rho) = coth\left(\frac{\rho h}{2}\right)$$

Therefore we define

$$\mu(\rho) = coth\left(\frac{\rho h}{2}\right) \text{ , when } a(x) = a = \text{constant}$$

and

$$\mu(\rho_j) = coth\left(\frac{\rho_j h}{2}\right) \text{ , when } a(x) \neq \text{constant}$$

Hence with the above $\mu(\rho_j)$, the variable fitting factor $\sigma_j$ is defined as:

$$\sigma_j = \frac{h\alpha_j}{2}\,\mu(\rho_j) \tag{4.27}$$

## 4.4 Proofs of the uniform convergence

Proofs of the uniform convergence for VMSC:

For the error analysis of the problem (4.1), when $b(x) \equiv 0$, we have used the comparison functions method (as in chapter 2).

This method uses the following two Lemmas [26]:

**Lemma 4.3** *(maximum principle) : Let* $\{u_j\}$ *be a set of values at the grid points* $x_j$*, satisfying* $u_0 \leq 0, u_n \leq 0$ *and* $Ru_j \geq 0, j = 1(1)n-1$*, then* $u_j \leq 0, j = 0(1)n$.

**Lemma 4.4** *If* $K_1(h,\varepsilon) \geq 0$ *and* $K_2(h,\varepsilon) \geq 0$ *are such that*

$$R(K_1(h,\varepsilon)\phi_j + K_2(h,\varepsilon)\psi_j) \geq R(\pm e_j) = \pm\tau_j(y)$$

*for each* $j = 1, 2, \cdots, n-1$*, then the discrete maximum principle implies that*

$$|e_j| \leq K_1(h,\varepsilon)|\phi_j| + K_2(h,\varepsilon)|\psi_j|$$

*where,* $|e_j| = |u_j - y(x_j)|$*, for each* $j$ *and* $\phi$ *and* $\psi$ *are two comparison functions.*

The following Lemma [26] gives the properties of the exact solution of (4.14):

**Lemma 4.5** *[26] Let* $a(x), f(x) \in C^3(0,1)$*, then*

$$y(x) = v(x) + w(x)$$

*where*

$$v(x) = \left(-\varepsilon\frac{y'(0)}{a(0)}\right) exp\left(-\frac{a(0)x}{\varepsilon}\right)$$

*and*

$$\left|w^{(k)}(x)\right| \leq M\left[1 + \varepsilon^{-k+1} exp\left(-\frac{\delta x}{\varepsilon}\right)\right]$$

$k = 0(1)4$ , $\delta = a/4$*, and* $0 < a < a(x)$ *for all* $x$ *and* $M$ *is a positive constant independent of* $h$ *and* $\varepsilon$.

We use two comparison functions (as in [26] ): $\phi = -2+x$ and $\psi = -exp(-\beta x/\varepsilon)$, ($\beta$ will be taken to be the smallest of various constants appearing in the proof). Therefore

$$\phi_j = -2 + x_j \quad \text{and} \quad \psi_j = -[\mu(\beta)]^j,\ j = 0(1)n$$

where

$$\mu(\beta) = [r^-(\beta h_c/\varepsilon)/r^+(\beta h_c/\varepsilon)] = exp(-\beta h_c/\varepsilon)$$

where

$$h_c = \max_{1\leq j\leq n} h_j \quad (= \text{a constant})$$

Remark :- The following inequalities hold for $h_j < 2\varepsilon/(a_{j-1}+a_j)$:

$$R\phi_j \geq Mh_c/\varepsilon$$

$$R\psi_j \geq M(\mu(\beta))^j h_c/\varepsilon^2$$

Now we estimate the truncation error of the scheme (4.18) using (4.19).

The local truncation error $\tau_j(\phi)$ of the scheme (4.18), is defined by

$$\tau_j(\phi) = R\phi_j - Q(L\phi)_j$$

where, $\phi(x)$ is an arbitrary sufficiently smooth function.

Therefore we have

$$\tau_j(y) = Ry_j - Q(Ly)_j$$

where

$$Ry_j = r_j^- y_{j-1} + r_j^c y_j + r_j^+ y_{j+1}$$

and

$$\begin{aligned} Q(Ly)_j &= q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1} \\ &= q_j^-(\varepsilon y''_{j-1} + a_{j-1}y'_{j-1}) + q_j^c(\varepsilon y''_j + a_j y'_j) \\ &\quad + q_j^+(\varepsilon y''_{j+1} + a_{j+1}y'_{j+1}) \end{aligned}$$

$$\Rightarrow \qquad \tau_j(y) = T_0 y_j + T_1 y'_j + T_2 y''_j + \text{Remainder terms}$$

where,

$$\begin{aligned} T_0 &= r_j^- + r_j^c + r_j^+ \\ T_1 &= \left(h_{j+1}r_j^+ - h_j r_j^-\right) - \left(q_j^- a_{j-1} + q_j^c a_j + q_j^+ a_{j+1}\right) \\ T_2 &= \tfrac{1}{2}\left(h_j^2 r_j^- + h_{j+1}^2 r_j^+\right) - \varepsilon\left(q_j^- + q_j^c + q_j^+\right) + \left(h_j q_j^- a_{j-1} - h_{j+1} q_j^+ a_{j+1}\right) \end{aligned}$$

Using (4.19) we see that $T_0 = 0, T_1 = 0$ and

$$|T_2| \leq M\frac{h_c^2}{\varepsilon}$$

Now using Lemma 4.5, we have

$$\left|w_j^{(2)}\right| \leq M\left[1+\frac{1}{\varepsilon}exp\left(-\frac{\delta x_j}{\varepsilon}\right)\right]$$

$$\Rightarrow |\tau_j(w)| \leq M\frac{h_c^2}{\varepsilon}\left[1+\frac{1}{\varepsilon}exp\left(-\frac{\delta x_j}{\varepsilon}\right)\right]$$

Also from Lemma 4.5, we have

$$v_j'' = -\frac{a(0)}{\varepsilon^2}v_j$$

therefore

$$|\tau_j(v)| \leq \frac{Mh_c^2}{\varepsilon^2}exp\left(-\frac{a(0)x_j}{\varepsilon}\right)$$

Since

$$\tau_j(y) = \tau_j(v) + \tau_j(w)$$

$$\Rightarrow \quad |\tau_j(y)| \leq M\left[\frac{h_c^2}{\varepsilon}+\frac{h_c^2}{\varepsilon^2}exp\left(-\frac{\delta x_j}{\varepsilon}\right)\right]$$

Choosing $K_1 = h_c^2$ and $K_2 = h_c^2/\varepsilon$, we see that Lemma 4.4 is satisfied and therefore we have the following theorem:

**Theorem 4.1** *Let $\{u_j\}$, $j = 0(1)n$, be a set of values of the approximate solution to $y(x)$ of (4.1), obtained using (4.18) and (4.19). Then there are positive constants $\beta$ and $M$ (independent of $h$ and $\varepsilon$) such that the following estimate holds:*

$$|y(x_j) - u_j| \leq Mh_c^2\left[1+\frac{1}{\varepsilon}exp\left(-\frac{\beta x_j}{\varepsilon}\right)\right]$$

*where,*

$$h_c = \max_{1\leq j\leq n} h_j \quad (= \textit{a constant})$$

Proofs of the uniform convergence for EFSC:

**CASE I**

The scheme (4.21), (4.22) can be written in the matrix form :

$$Au = F$$

where, A is a matrix of the system (4.21), u and F are corresponding vectors.

Now, the local truncation $\tau_j(\phi)$ of the scheme (4.21), is defined by

$$\tau_j(\phi) = R\phi_j - Q(L\phi)_j$$

where, $\phi(x)$ is an arbitrary sufficiently smooth function. Therefore,

$$\begin{aligned}
\tau_j(y) &= Ry_j - Q(Ly)_j \\
&= R(y_j - u_j) \\
\Rightarrow \quad R(y_j - u_j) &= \tau_j(y) \\
\Rightarrow \max_j |y_j - u_j| &\leq \|A^{-1}\| \max_j |\tau_j(y)| \qquad (4.28)
\end{aligned}$$

In order to estimate the values $|y_j - u_j|$, we will estimate the truncation error $\tau_j(y)$ and the norm of the matrix $A^{-1}$.

From (4.26), we see that

$$\sigma_j - \varepsilon = -\frac{h^2\beta_j}{4} + \varepsilon\left[\frac{\left\{h\left(\sqrt{\beta_j/\varepsilon}\right)/2\right\}^2}{sin^2\left\{h\left(\sqrt{\beta_j/\varepsilon}\right)/2\right\}} - 1\right]$$

$$\Rightarrow |\sigma_j - \varepsilon| \leq Mh^2 \qquad (4.29)$$

i.e. $\sigma_j$ approximates $\varepsilon$ with the error $O(h^2)$.

<u>Estimation of truncation error and the norm of $A^{-1}$</u>:-

From Lemma 4.1, we have

$$\tau_j(y) = \tau_j(v) + \tau_j(w) + \tau_j(g)$$

We will estimate separately the parts of $\tau_j(y)$.

We will start with $v(x)$:-

$$Rv_j = r_j^- v_{j-1} + r_j^c v_j + r_j^+ v_{j+1}$$

Expanding $v_{j-1}$ and $v_{j+1}$ in terms of $v_j$, we obtain

$$\begin{aligned}
Rv_j &= v_j\left[r_j^- exp\left(ih\sqrt{b_0/\varepsilon}\right) + r_j^c + r_j^+ exp\left(-ih\sqrt{b_0/\varepsilon}\right)\right] \\
&= v_j\left[-2 + 2cosh\left(h\sqrt{b_0/\varepsilon}\right) + \left\{h^2\beta_j/4\sigma_j\right\} exp\left(ih\sqrt{b_0/\varepsilon}\right)\right. \\
&\quad \left. + \left\{h^2\beta_{j+1}/4\sigma_{j+1}\right\} exp\left(-ih\sqrt{b_0/\varepsilon}\right) + (h^2/4)\left\{(\beta_j/\sigma_j) + (\beta_{j+1}/\sigma_{j+1})\right\}\right]
\end{aligned}$$

Expanding exponentials and using $\sigma_j = \varepsilon + O(h^2)$ (from (4.29)), we have

$$Rv_j = \frac{h^2}{\varepsilon}(b_j - b_0)v_j + O\left(\frac{h^4}{\varepsilon}\right) \tag{4.30}$$

and

$$\begin{aligned} Q(Lv)_j &= q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1} \\ &= q_j^-(\varepsilon v''_{j-1} + b_{j-1}v_{j-1}) + q_j^c(\varepsilon v''_j + b_j v_j) \\ &\quad + q_j^+(\varepsilon v''_{j+1} + b_{j+1}v_{j+1}) \end{aligned}$$

Expanding $v_{j-1}$ and $v_{j+1}$ and their derivatives in terms $v_j$ and its derivatives and using Lemma 4.1, we have

$$Q(Lv)_j = v_j\left[q_j^-(b_{j-1} - b_0)\,exp\left(ih\sqrt{b_0/\varepsilon}\right) + q_j^c(b_j - b_0) + q_j^+(b_{j+1} - b_0)\,exp\left(-ih\sqrt{b_0/\varepsilon}\right)\right]$$

Expanding exponentials and using (4.22) and (4.29), we have

$$Q(Lv)_j = \frac{h^2}{\varepsilon}(b_j - b_0)v_j + O\left(\frac{h^4}{\varepsilon}\right) \tag{4.31}$$

From equations (4.30) and (4.31), we have

$$|\tau_j(v)| = |Rv_j - Q(Lv)_j| \leq Mh^4/\varepsilon \tag{4.32}$$

Similarly,

$$Rw_j = \frac{h^2}{\varepsilon}(b_j - b_1)w_j + O\left(\frac{h^4}{\varepsilon}\right) \tag{4.33}$$

$$Q(Lw)_j = \frac{h^2}{\varepsilon}(b_j - b_1)w_j + O\left(\frac{h^4}{\varepsilon}\right) \tag{4.34}$$

$$\Rightarrow \quad |\tau_j(w)| \leq Mh^4/\varepsilon \tag{4.35}$$

Now

$$\tau_j(g) = Rg_j - Q(Lg)_j \tag{4.36}$$

where

$$\begin{aligned} Rg_j &= r_j^- g_{j-1} + r_j^c g_j + r_j^+ g_{j+1} \\ &= (r_j^- + r_j^c + r_j^+)g_j + (r_j^+ - r_j^-)hg'_j + (r_j^+ + r_j^-)\frac{h^2}{2!}g''_j + \cdots \end{aligned}$$

and

$$\begin{aligned} Q(Lg)_j &= q_j^-(\varepsilon g''_{j-1} + b_{j-1}g_{j-1}) + q_j^c(\varepsilon g''_j + b_j g_j) + q_j^+(\varepsilon g''_{j+1} + b_{j+1}g_{j+1}) \\ &= \left[q_j^- b_{j-1} + q_j^c b_j + q_j^+ b_{j+1}\right] g_j + \left[h(-q_j^- b_{j-1} + q_j^+ b_{j+1})\right] g'_j \\ &+ \left[q_j^-(\varepsilon + \frac{h^2}{2} b_{j-1}) + q_j^c \varepsilon + q_j^+(\varepsilon + \frac{h^2}{2} b_{j+1})\right] g''_j + \cdots \end{aligned}$$

Therefore from (4.36), we have

$$\tau_j(g) = T_0 g_j + T_1 g'_j + \text{ Remainder terms}$$

where

$$T_0 = (r_j^- + r_j^c + r_j^+) - (q_j^- b_{j-1} + q_j^c b_j + q_j^+ b_{j+1})$$
$$T_1 = (r_j^+ - r_j^-)h - (q_j^+ b_{j+1} - q_j^- b_{j-1})h$$

Using (4.22) we see that $T_0 = 0$ and

$$T_1 = \frac{h^3}{4}\left[\frac{\beta_{j+1} - b_{j+1}}{\sigma_{j+1}} - \frac{\beta_j - b_j}{\sigma_j}\right]$$

Therefore using (4.29), we get $|T_1| \le Mh^4/\varepsilon$. Hence

$$\begin{aligned} |T_1 g'_j| &\le (Mh^4/\varepsilon)|g'_j| \\ &\le Mh^4/\varepsilon \quad \text{(using Lemma 4.1)} \\ \Rightarrow \quad |\tau_j(g)| &\le Mh^4/\varepsilon \end{aligned} \tag{4.37}$$

From (4.32), (4.35) and (4.37), we have

$$|\tau_j(y)| \le Mh^4/\varepsilon \tag{4.38}$$

<u>Estimate of $\|A^{-1}\|$</u> :-

Since $r_j^c < 0$ and $r_j^\pm > 0$, therefore

$$\|A^{-1}\| \le \max_j |r_j^- + r_j^c + r_j^+|^{-1}$$ (Varah [248])

Now

$$\begin{aligned}
|r_j^- + r_j^c + r_j^+| &= |r_j^- + (-4 + r_j^- + r_j^+) + r_j^+| \\
&= \left|\frac{h^2}{2}\left(\frac{\beta_j}{\sigma_j} + \frac{\beta_{j+1}}{\sigma_{j+1}}\right)\right| \\
&\geq M_1 \frac{h^2}{\varepsilon} \quad \text{using (4.26)} \\
\Rightarrow \quad |r_j^- + r_j^c + r_j^+|^{-1} &\leq M\frac{\varepsilon}{h^2} \quad \left(\text{where, } M = \frac{1}{M_1}\right) \\
\Rightarrow \max_j |r_j^- + r_j^c + r_j^+|^{-1} &\leq M\frac{\varepsilon}{h^2} \\
\Rightarrow \quad \|A^{-1}\| &\leq M\frac{\varepsilon}{h^2} \qquad (4.39)
\end{aligned}$$

Hence from (4.28), (4.38) and (4.39), we have the following theorem:

**Theorem 4.2** *Let $b(x)$, $f(x) \in C^2[0,1]$ and $b(x) \geq b > 0, b'(0) = b'(1) = 0$. Let $u_j$, j=0(1)n, be the approximate solution of (4.1), when $a(x) \equiv 0$, obtained using (4.21), (4.22). Then, there is a constant $M$ independent of $\varepsilon$ and $h$ such that*

$$\max_j |y_j - u_j| \leq Mh^2$$

## CASE II

For the error analysis in this case we have used the comparison functions method developed by Kellogg and Tsan [131] and Berger et al. [26]. By a comparison function we mean a function $\phi$ such that $L\phi_i > 0$, $-N < i < N$ and $\phi_{\pm N} > 0$, where $L$ is a differential operator and $N$ is a positive integer. These functions are used together with the maximum principle to convert the bounds on truncation error to bounds on descritization error.

This method uses the following two Lemmas [26] :

**Lemma 4.6** *(maximum principle) : Let $\{u_j\}$ be a set of values at the grid points $x_j$, satisfying $u_0 \leq 0, u_n \leq 0$ and $Ru_j \geq 0, j = 1(1)n-1$, then $u_j \leq 0, j = 0(1)n$.*

**Lemma 4.7** *If $K_1(h,\varepsilon) \geq 0$ and $K_2(h,\varepsilon) \geq 0$ are such that*

$$R(K_1(h,\varepsilon)\phi_j + K_2(h,\varepsilon)\psi_j) \geq R(\pm e_j) = \pm\tau_j(y)$$

*for each $j = 1, 2, \cdots, n-1$, then the discrete maximum principle implies that*

$$|e_j| \;\leq\; K_1(h,\varepsilon)|\phi_j| \;+\; K_2(h,\varepsilon)|\psi_j|$$

*where, $|e_j| \;=\; |u_j - y(x_j)|$, for each $j$ and $\phi$ and $\psi$ are two comparison functions.*

We use two comparison functions (as in [26] ) : $\phi = -2 + x$ and $\psi = -exp(-\beta x/\varepsilon)$, ($\beta$ will be taken to be the smallest of various constants appearing in the proof). Therefore $\phi_j = -2 + x_j$ and $\psi_j = -[\mu(\beta)]^j$, $j = 0(1)n$; where, $\mu(\beta) = [r^-(\beta h/\varepsilon)/r^+(\beta h/\varepsilon)] = exp(-\beta h/\varepsilon)$.

From (4.27), we see that

$$\sigma_j - \varepsilon = \varepsilon\left[\frac{h\alpha_j}{2\varepsilon}\,coth\left(\frac{h\alpha_j}{2\varepsilon}\right) - 1\right]$$

Using $x\,coth\,x = 1 + \frac{x^2}{3} + O(x^4)$ and the fact that the consistency contition for this case requires $h < \varepsilon/\alpha_j$, we get

$$|\sigma_j - \varepsilon| \leq Mh \tag{4.40}$$

i.e. $\sigma_j$ approximates $\varepsilon$ with the error $O(h)$.

<u>Remark</u> :- Using (4.40), we see that the following inequalities hold for $h < \varepsilon/\alpha_j$ :

$$R\phi_j \geq Mh^2/\varepsilon \;\;,\;\; R\psi_j \geq M(\mu(\beta))^j h^2/\varepsilon^2$$

Now we estimate the truncation error of the scheme (4.21) using (4.23).

We have,

$$\tau_j(y) \;=\; T_0 y_j + T_1 y_j' + T_2 y_j'' + \text{Remainder terms}$$

where,

$$T_0 \;=\; r_j^- + r_j^c + r_j^+$$
$$T_1 \;=\; h(r_j^+ - r_j^-) - (q_j^- a_{j-1} + q_j^c a_j + q_j^+ a_{j+1})$$

$$T_2 \;=\; \tfrac{h^2}{2}(r_j^- + r_j^+) - \varepsilon(q_j^- + q_j^c + q_j^+) + h(q_j^- a_{j-1} - q_j^+ a_{j+1})$$

Using (4.23) we see that $T_0 = 0, T_1 = 0$ and

$$T_2 = \frac{h^2}{2}\left[\left(1 - \frac{h\alpha_j}{2\sigma_j}\right) + \left(1 + \frac{h\alpha_{j+1}}{2\sigma_{j+1}}\right)\right] - \frac{\varepsilon h^2}{2}\left(\frac{1}{\sigma_j} + \frac{1}{\sigma_{j+1}}\right) + \frac{h^3}{4}\left(\frac{a_{j-1}}{\sigma_j} - \frac{a_{j+1}}{\sigma_{j+1}}\right)$$

Using $\sigma_j = \varepsilon + O(h)$ (from (4.40), we get

$$|T_2| \leq Mh^3/\varepsilon$$

Therefore using Lemma 4.2, we have

$$\left|T_2 w_j^{(2)}\right| \;\leq\; M\frac{h^3}{\varepsilon}\left[1 + \frac{1}{\varepsilon}exp\left(-\frac{\delta x_j}{\varepsilon}\right)\right]$$

Also from Lemma 4.2, we have $v_j'' = (-a(0)/\varepsilon)^2 v_j$, therefore

$$|\tau_j(v)| \;\leq\; \frac{Mh^3}{\varepsilon^2}exp\left(-\frac{a(0)x_j}{\varepsilon}\right)$$

$$\Rightarrow\quad |\tau_j(y)| \;\leq\; M\left[\frac{h^3}{\varepsilon} + \frac{h^3}{\varepsilon^2}exp\left(-\frac{\delta x_j}{\varepsilon}\right)\right]$$

Choosing $K_1 = h^2$ and $K_2 = h^2/\varepsilon$, we see that Lemma 4.7 is satisfied and therefore we have the following theorem:

**Theorem 4.3** *Let $\{u_j\}$, $j = 0(1)n$, be a set of values of the approximate solution to $y(x)$ of (4.1), when $b(x) \equiv 0$, obtained using (4.21) and (4.23). Then there are positive constants $\beta$ and $M$ (independent of $h$ and $\varepsilon$) such that the following estimate holds:*

$$|y(x_j) - u_j| \;\leq\; Mh^2\left[1 + \frac{1}{\varepsilon}exp\left(-\frac{\beta x_j}{\varepsilon}\right)\right]$$

# 4.5 Test Examples and Numerical Results

To illustrate the predicted theory, we solve the following problems:

**Example 4.1** [24] : *Consider $\varepsilon y'' + y = 0$ ; $y(0) = 0$ , $y(1) = 1$, whose exact solution is given by*

$$y(x) = \frac{sin(x/\sqrt{\varepsilon})}{sin(1/\sqrt{\varepsilon})} \;,\quad \varepsilon \neq (n\pi)^{-2}$$

**Example 4.2** [45] : *Consider* $\varepsilon y'' + (\pi^2/4)y = 0$ ; $y(0) = 0$ , $y(1) = sin\,(\pi/2\sqrt{\varepsilon})$, *whose exact solution is given by*

$$y(x) = sin\,(\pi x/2\sqrt{\varepsilon})$$

**Example 4.3** [145] : *Consider* $\varepsilon y'' + 3/\{(1 + (x^2/\varepsilon)^2\}y = 0$ ; $y(0) = 0$ , $y(0.1) = 0.1/\sqrt{\varepsilon + 0.01}$, *whose exact solution is given by*

$$y(x) = x/\sqrt{\varepsilon + x^2}$$

**Example 4.4** [135] : *Consider* $\varepsilon y'' + y' = 2$ ; $y(0) = 0$ , $y(1) = 1$, *whose exact solution is given by*

$$y(x) = 2x + \frac{1 - e^{-(x/\varepsilon)}}{e^{-(1/\varepsilon)} - 1}$$

**Example 4.5** [209] : *Consider*

$$\varepsilon y'' + (x+1)^3 y' = f(x)\ ;\ y(0) = 2\ ,\ y(1) = \left(\frac{1}{8}\right) exp\left(-\frac{15}{4\varepsilon}\right) + exp\left(-\frac{1}{2}\right)$$

*whose exact solution is given by*

$$y(x) = \frac{1}{(x+1)^3} exp\left[-\frac{1}{4\varepsilon}\{(x+1)^4 - 1\}\right] + exp\left(-\frac{x}{2}\right)$$

**Example 4.6** [249] : *Consider* $\varepsilon y'' + \left[\frac{2\varepsilon}{1+x} + \frac{2}{(1+x)^2}\right] y' = 0$ ; $y(0) = 0$ , $y(1) = 0$, *whose exact solution is given by*

$$y(x) = cos\left(\frac{\pi x}{1+x}\right) + \frac{exp(-1/\varepsilon) - exp(-2x/(\varepsilon(1+x)))}{1 - exp(-1/\varepsilon)}$$

Tables 4.1 - 4.8 and 4.11 - 4.23 except Table 4.21 contain the maximum errors at all the mesh points:

$$\max_j |y(x_j) - u_j| \tag{4.41}$$

for different $n$ and $\varepsilon$.

Tables 4.9, 4.10 and 4.21 contain the numerical rate of uniform convergence which is determined as in [66]:

$$r_{k,\varepsilon} = log_2\left(\frac{z_{k,\varepsilon}}{z_{k+1,\varepsilon}}\right)\ ,\ k = 0, 1, \cdots\cdots\cdots \tag{4.42}$$

where,

$$z_{k,\varepsilon} = \max_j |u_j^{h_j/2^k} - u_{2j}^{h_j/2^{k+1}}|\ ,\ k = 0, 1, \cdots\cdots\cdots$$

and $u_j^{h_j/2^k}$ denotes the value of $u_j$ for the mesh length $h_j/2^k$.

Table 4.1: Max. Errors for Example 4.4
With Uniform Mesh

| $\varepsilon$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ | $n=512$ |
|---|---|---|---|---|---|---|
| 1/16 | 0.35E-01 | 0.79E-02 | 0.19E-02 | 0.48E-03 | 0.12E-03 | 0.30E-04 |
| 1/32 | 0.14E+00 | 0.35E-01 | 0.79E-02 | 0.19E-02 | 0.48E-03 | 0.12E-03 |
| 1/64 | 0.35E+00 | 0.14E+00 | 0.35E-01 | 0.79E-02 | 0.19E-02 | 0.48E-03 |
| 1/128 | 0.60E+00 | 0.35E+00 | 0.14E+00 | 0.35E-01 | 0.79E-02 | 0.19E-02 |
| 1/256 | 0.81E+00 | 0.60E+00 | 0.35E+00 | 0.14E+00 | 0.35E-01 | 0.79E-02 |
| 1/512 | 0.12E+01 | 0.78E+00 | 0.60E+00 | 0.35E+00 | 0.14E+00 | 0.35E-01 |
| 1/1024 | 0.21E+01 | 0.92E+00 | 0.78E+00 | 0.60E+00 | 0.35E+00 | 0.14E+00 |

Table 4.2: Max. Errors for Example 4.4
Using VMSC: With about 12.5% mesh points in the boundary layer region

| $\varepsilon$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ | $n=512$ |
|---|---|---|---|---|---|---|
| 1/16 | 0.98E-01 | 0.23E-01 | 0.53E-02 | 0.13E-02 | 0.33E-03 | 0.83E-04 |
| 1/32 | 0.14E+00 | 0.38E-01 | 0.85E-02 | 0.21E-02 | 0.52E-03 | 0.13E-03 |
| 1/64 | 0.19E+00 | 0.56E-01 | 0.12E-01 | 0.30E-02 | 0.74E-03 | 0.19E-03 |
| 1/128 | 0.24E+00 | 0.76E-01 | 0.17E-01 | 0.41E-02 | 0.10E-02 | 0.25E-03 |
| 1/256 | 0.29E+00 | 0.98E-01 | 0.23E-01 | 0.53E-02 | 0.13E-02 | 0.33E-03 |
| 1/512 | 0.35E+00 | 0.12E+00 | 0.30E-01 | 0.68E-02 | 0.17E-02 | 0.41E-03 |
| 1/1024 | 0.45E+00 | 0.14E+00 | 0.37E-01 | 0.84E-02 | 0.20E-02 | 0.49E-03 |

Table 4.3: Max. Errors for Example 4.4
Using VMSC: With about 25% mesh points in the boundary layer region

| $\varepsilon$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ | $n=512$ |
|---|---|---|---|---|---|---|
| 1/16 | 0.23E-01 | 0.53E-02 | 0.13E-02 | 0.33E-03 | 0.83E-04 | 0.21E-04 |
| 1/32 | 0.38E-01 | 0.85E-02 | 0.21E-02 | 0.52E-03 | 0.13E-03 | 0.32E-04 |
| 1/64 | 0.56E-01 | 0.12E-01 | 0.30E-02 | 0.74E-03 | 0.19E-03 | 0.46E-04 |
| 1/128 | 0.76E-01 | 0.17E-01 | 0.41E-02 | 0.10E-02 | 0.25E-03 | 0.63E-04 |
| 1/256 | 0.98E-01 | 0.23E-01 | 0.53E-02 | 0.13E-02 | 0.33E-03 | 0.81E-04 |
| 1/512 | 0.12E+00 | 0.30E-01 | 0.68E-02 | 0.17E-02 | 0.41E-03 | 0.99E-04 |
| 1/1024 | 0.14E+00 | 0.37E-01 | 0.84E-02 | 0.20E-02 | 0.49E-03 | 0.11E-03 |

Table 4.4: Max. Errors for Example 4.4
Using VMSC: With about 50% mesh points in the boundary layer region

| $\varepsilon$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ | $n=512$ |
|---|---|---|---|---|---|---|
| 1/16 | 0.53E-02 | 0.13E-02 | 0.33E-03 | 0.83E-04 | 0.21E-04 | 0.52E-05 |
| 1/32 | 0.85E-02 | 0.21E-02 | 0.52E-03 | 0.13E-03 | 0.32E-04 | 0.80E-05 |
| 1/64 | 0.12E-01 | 0.30E-02 | 0.80E-03 | 0.20E-03 | 0.47E-04 | 0.12E-04 |
| 1/128 | 0.17E-01 | 0.41E-02 | 0.10E-02 | 0.37E-03 | 0.93E-04 | 0.21E-04 |
| 1/256 | 0.23E-01 | 0.53E-02 | 0.13E-02 | 0.44E-03 | 0.17E-03 | 0.42E-04 |
| 1/512 | 0.30E-01 | 0.69E-02 | 0.17E-02 | 0.41E-03 | 0.19E-03 | 0.74E-04 |
| 1/1024 | 0.37E-01 | 0.85E-02 | 0.20E-02 | 0.49E-03 | 0.15E-03 | 0.85E-04 |

Table 4.5: Max. Errors for Example 4.5
With Uniform Mesh

| $\varepsilon$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ | $n=512$ |
|---|---|---|---|---|---|---|
| 1/16 | 0.42E-01 | 0.94E-02 | 0.23E-02 | 0.57E-03 | 0.14E-03 | 0.36E-04 |
| 1/32 | 0.14E+00 | 0.37E-01 | 0.83E-02 | 0.20E-02 | 0.51E-03 | 0.13E-03 |
| 1/64 | 0.33E+00 | 0.14E+00 | 0.35E-01 | 0.80E-02 | 0.20E-02 | 0.49E-03 |
| 1/128 | 0.54E+00 | 0.34E+00 | 0.13E+00 | 0.35E-01 | 0.79E-02 | 0.19E-02 |
| 1/256 | 0.72E+00 | 0.56E+00 | 0.34E+00 | 0.14E+00 | 0.35E-01 | 0.79E-02 |
| 1/512 | 0.86E+00 | 0.73E+00 | 0.58E+00 | 0.35E+00 | 0.14E+00 | 0.35E-01 |
| 1/1024 | 0.96E+00 | 0.86E+00 | 0.75E+00 | 0.59E+00 | 0.35E+00 | 0.14E+00 |

Table 4.6: Max. Errors for Example 4.5
Using VMSC: With about 12.5% mesh points in the boundary layer region

| $\varepsilon$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ | $n=512$ |
|---|---|---|---|---|---|---|
| 1/16 | 0.82E-01 | 0.18E-01 | 0.44E-02 | 0.11E-02 | 0.27E-03 | 0.68E-04 |
| 1/32 | 0.11E+00 | 0.27E-01 | 0.62E-02 | 0.15E-02 | 0.38E-03 | 0.95E-04 |
| 1/64 | 0.14E+00 | 0.38E-01 | 0.87E-02 | 0.21E-02 | 0.53E-03 | 0.13E-03 |
| 1/128 | 0.18E+00 | 0.53E-01 | 0.12E-01 | 0.28E-02 | 0.71E-03 | 0.18E-03 |
| 1/256 | 0.22E+00 | 0.69E-01 | 0.15E-01 | 0.37E-02 | 0.92E-03 | 0.23E-03 |
| 1/512 | 0.27E+00 | 0.86E-01 | 0.19E-01 | 0.46E-02 | 0.12E-02 | 0.28E-03 |
| 1/1024 | 0.31E+00 | 0.10E+00 | 0.25E-01 | 0.57E-02 | 0.14E-02 | 0.34E-03 |

Table 4.7: Max. Errors for Example 4.5
Using VMSC: With about 25% mesh points in the boundary layer region

| $\varepsilon$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ | $n=512$ |
|---|---|---|---|---|---|---|
| 1/16 | 0.18E-01 | 0.45E-02 | 0.11E-02 | 0.28E-03 | 0.69E-04 | 0.17E-04 |
| 1/32 | 0.27E-01 | 0.62E-02 | 0.15E-02 | 0.38E-03 | 0.96E-04 | 0.24E-04 |
| 1/64 | 0.39E-01 | 0.87E-02 | 0.21E-02 | 0.53E-03 | 0.13E-03 | 0.33E-04 |
| 1/128 | 0.53E-01 | 0.12E-01 | 0.29E-02 | 0.71E-03 | 0.18E-03 | 0.44E-04 |
| 1/256 | 0.69E-01 | 0.15E-01 | 0.37E-02 | 0.92E-03 | 0.23E-03 | 0.56E-04 |
| 1/512 | 0.87E-01 | 0.20E-01 | 0.47E-02 | 0.12E-02 | 0.37E-03 | 0.11E-03 |
| 1/1024 | 0.11E+00 | 0.25E-01 | 0.58E-02 | 0.14E-02 | 0.41E-03 | 0.19E-03 |

Table 4.8: Max. Errors for Example 4.5
Using VMSC: With about 50% mesh points in the boundary layer region

| $\varepsilon$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ | $n=512$ |
|---|---|---|---|---|---|---|
| 1/16 | 0.51E-02 | 0.14E-02 | 0.34E-03 | 0.82E-04 | 0.21E-04 | 0.51E-05 |
| 1/32 | 0.70E-02 | 0.25E-02 | 0.75E-03 | 0.17E-03 | 0.42E-04 | 0.11E-04 |
| 1/64 | 0.98E-02 | 0.33E-02 | 0.15E-02 | 0.41E-03 | 0.92E-04 | 0.23E-04 |
| 1/128 | 0.13E-01 | 0.31E-02 | 0.20E-02 | 0.82E-03 | 0.22E-03 | 0.50E-04 |
| 1/256 | 0.16E-01 | 0.40E-02 | 0.18E-02 | 0.11E-02 | 0.45E-03 | 0.12E-03 |
| 1/512 | 0.20E-01 | 0.50E-02 | 0.13E-02 | 0.10E-02 | 0.60E-03 | 0.24E-03 |
| 1/1024 | 0.26E-01 | 0.60E-02 | 0.15E-02 | 0.69E-03 | 0.53E-03 | 0.32E-03 |

Table 4.9: Rate of convergence for Example 4.4
Using VMSC: With about 25% mesh points in the boundary layer region
$n = 16, 32, 64, 128, 256$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/2 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/4 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.19E+01 | 0.20E+01 |
| 1/8 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.19E+01 | 0.20E+01 |
| 1/16 | 0.23E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.21E+01 |
| 1/32 | 0.19E+01 | 0.23E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/64 | —— | 0.19E+01 | 0.23E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/128 | —— | —— | 0.19E+01 | 0.23E+01 | 0.20E+01 | 0.20E+01 |

Table 4.10: Rate of convergence for Example 4.5
Using VMSC: With about 25% mesh points in the boundary layer region
$n = 16, 32, 64, 128, 256$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/2 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/4 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.18E+01 | 0.20E+01 |
| 1/8 | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.17E+01 | 0.20E+01 |
| 1/16 | 0.22E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.21E+01 |
| 1/32 | 0.18E+01 | 0.23E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/64 | —— | 0.18E+01 | 0.23E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/128 | —— | —— | 0.18E+01 | 0.23E+01 | 0.20E+01 | 0.20E+01 |

Table 4.11: Max. Errors for Example 4.1
Without Using Fitting Factor

| $\varepsilon$ | n = 16 | n = 32 | n = 64 | n = 128 | n = 256 | n = 512 | n = 1024 |
|---|---|---|---|---|---|---|---|
| 1/8 | 0.73E-01 | 0.19E-01 | 0.47E-02 | 0.12E-02 | 0.29E-03 | 0.73E-04 | 0.18E-04 |
| 1/16 | 0.29E-01 | 0.71E-02 | 0.18E-02 | 0.44E-03 | 0.11E-03 | 0.28E-04 | 0.69E-05 |
| 1/32 | 0.14E+00 | 0.38E-01 | 0.98E-02 | 0.25E-02 | 0.62E-03 | 0.15E-03 | 0.39E-04 |
| 1/64 | 0.13E+00 | 0.34E-01 | 0.87E-02 | 0.22E-02 | 0.55E-03 | 0.14E-03 | 0.34E-04 |
| 1/128 | 0.39E+00 | 0.11E+00 | 0.28E-01 | 0.72E-02 | 0.18E-02 | 0.45E-03 | 0.11E-03 |
| 1/256 | 0.46E+01 | 0.38E+02 | 0.14E+01 | 0.27E+00 | 0.63E-01 | 0.16E-01 | 0.39E-02 |
| 1/512 | 0.29E+01 | 0.58E+01 | 0.95E+00 | 0.18E+00 | 0.42E-01 | 0.10E-01 | 0.25E-02 |

Table 4.12: Max. Errors for Example 4.1
Using Fitting Factor

| $\varepsilon$ | n = 16 | n = 32 | n = 64 | n = 128 | n = 256 | n = 512 | n = 1024 |
|---|---|---|---|---|---|---|---|
| 1/8 | 0.89E-13 | 0.57E-12 | 0.45E-11 | 0.93E-11 | 0.22E-10 | 0.67E-10 | 0.96E-09 |
| 1/16 | 0.13E-13 | 0.10E-13 | 0.61E-13 | 0.38E-12 | 0.47E-11 | 0.16E-11 | 0.76E-10 |
| 1/32 | 0.64E-14 | 0.32E-13 | 0.32E-12 | 0.23E-11 | 0.50E-11 | 0.12E-10 | 0.36E-10 |
| 1/64 | 0.65E-14 | 0.16E-13 | 0.88E-14 | 0.69E-13 | 0.48E-12 | 0.58E-11 | 0.15E-11 |
| 1/128 | 0.21E-14 | 0.42E-14 | 0.21E-13 | 0.20E-12 | 0.17E-11 | 0.39E-11 | 0.84E-11 |
| 1/256 | 0.10E-13 | 0.21E-12 | 0.39E-12 | 0.23E-12 | 0.23E-11 | 0.13E-10 | 0.17E-09 |
| 1/512 | 0.18E-13 | 0.63E-14 | 0.25E-13 | 0.13E-12 | 0.11E-11 | 0.94E-11 | 0.22E-10 |

Table 4.13: Max. Errors for Example 4.2
Without Using Fitting Factor

| $\varepsilon = 2^{-k}$ | n = 16 | n = 32 | n = 64 | n = 128 | n = 256 | n = 512 | n = 1024 |
|---|---|---|---|---|---|---|---|
| $k = 1$ | 0.32E-02 | 0.81E-03 | 0.20E-03 | 0.51E-04 | 0.13E-04 | 0.32E-05 | 0.79E-06 |
| $k = 3$ | 0.20E-01 | 0.51E-02 | 0.13E-02 | 0.32E-03 | 0.79E-04 | 0.20E-04 | 0.49E-05 |
| $k = 5$ | 0.29E+00 | 0.96E-01 | 0.26E-01 | 0.66E-02 | 0.17E-02 | 0.42E-03 | 0.10E-03 |
| $k = 7$ | 0.17E+01 | 0.39E+00 | 0.11E+00 | 0.30E-01 | 0.75E-02 | 0.19E-02 | 0.47E-03 |
| $k = 9$ | 0.18E+01 | 0.17E+01 | 0.68E+01 | 0.32E+00 | 0.69E-01 | 0.17E-01 | 0.42E-02 |
| $k = 11$ | 0.20E+01 | 0.52E+01 | 0.18E+01 | 0.44E+01 | 0.43E+00 | 0.12E+00 | 0.30E-01 |
| $k = 13$ | 0.12E+01 | 0.17E+01 | 0.27E+01 | 0.17E+01 | 0.26E+01 | 0.80E+01 | 0.41E+00 |
| $k = 15$ | 0.33E+01 | 0.20E+01 | 0.24E+01 | 0.23E+01 | 0.20E+01 | 0.23E+01 | 0.59E+01 |

Table 4.14: Max. Errors for Example 4.2
Using Fitting Factor

| $\varepsilon = 2^{-k}$ | n = 16 | n = 32 | n = 64 | n = 128 | n = 256 | n = 512 | n = 1024 |
|---|---|---|---|---|---|---|---|
| $k = 1$ | 0.16E-13 | 0.77E-13 | 0.21E-12 | 0.37E-13 | 0.11E-11 | 0.15E-10 | 0.34E-11 |
| $k = 3$ | 0.59E-14 | 0.24E-13 | 0.11E-12 | 0.35E-12 | 0.31E-13 | 0.17E-11 | 0.23E-10 |
| $k = 5$ | 0.14E-14 | 0.26E-13 | 0.12E-12 | 0.60E-12 | 0.17E-11 | 0.99E-13 | 0.98E-11 |
| $k = 7$ | 0.34E-14 | 0.18E-14 | 0.27E-13 | 0.14E-12 | 0.71E-12 | 0.18E-11 | 0.27E-12 |
| $k = 9$ | 0.46E-14 | 0.99E-14 | 0.41E-14 | 0.60E-13 | 0.29E-12 | 0.16E-11 | 0.41E-11 |
| $k = 11$ | 0.64E-14 | 0.13E-13 | 0.17E-13 | 0.11E-13 | 0.11E-12 | 0.54E-12 | 0.29E-11 |
| $k = 13$ | 0.20E-13 | 0.19E-13 | 0.23E-13 | 0.60E-13 | 0.20E-13 | 0.28E-12 | 0.14E-11 |
| $k = 15$ | 0.40E-13 | 0.41E-13 | 0.33E-13 | 0.51E-13 | 0.81E-13 | 0.34E-13 | 0.42E-12 |

Table 4.15: Max. Errors for Example 4.3
Without Using Fitting Factor

| $\varepsilon$ | n = 16 | n = 32 | n = 64 | n = 128 | n = 256 | n = 512 | n = 1024 |
|---|---|---|---|---|---|---|---|
| 1/2 | 0.95E-06 | 0.24E-06 | 0.61E-07 | 0.15E-07 | 0.38E-08 | 0.95E-09 | 0.24E-09 |
| 1/4 | 0.51E-05 | 0.13E-05 | 0.33E-06 | 0.82E-07 | 0.21E-07 | 0.51E-08 | 0.13E-08 |
| 1/8 | 0.27E-04 | 0.67E-05 | 0.17E-05 | 0.42E-06 | 0.11E-06 | 0.27E-07 | 0.66E-08 |
| 1/16 | 0.13E-03 | 0.32E-04 | 0.81E-05 | 0.20E-05 | 0.51E-06 | 0.13E-06 | 0.32E-07 |
| 1/32 | 0.53E-03 | 0.13E-03 | 0.34E-04 | 0.84E-05 | 0.21E-05 | 0.53E-06 | 0.13E-06 |
| 1/64 | 0.18E-02 | 0.44E-03 | 0.11E-03 | 0.28E-04 | 0.71E-05 | 0.18E-05 | 0.44E-06 |
| 1/128 | 0.42E-02 | 0.12E-02 | 0.29E-03 | 0.73E-04 | 0.18E-04 | 0.46E-05 | 0.11E-05 |

Table 4.16: Max. Errors for Example 4.3
Using Fitting Factor

| $\varepsilon$ | n = 16 | n = 32 | n = 64 | n = 128 | n = 256 | n = 512 | n = 1024 |
|---|---|---|---|---|---|---|---|
| 1/2 | 0.63E-06 | 0.16E-06 | 0.40E-07 | 0.10E-07 | 0.25E-08 | 0.63E-09 | 0.16E-09 |
| 1/4 | 0.34E-05 | 0.85E-06 | 0.22E-06 | 0.54E-07 | 0.13E-07 | 0.34E-08 | 0.84E-09 |
| 1/8 | 0.17E-04 | 0.43E-05 | 0.11E-05 | 0.27E-06 | 0.68E-07 | 0.17E-07 | 0.43E-08 |
| 1/16 | 0.80E-04 | 0.20E-04 | 0.50E-05 | 0.13E-05 | 0.31E-06 | 0.78E-07 | 0.20E-07 |
| 1/32 | 0.31E-03 | 0.77E-04 | 0.19E-04 | 0.48E-05 | 0.12E-05 | 0.30E-06 | 0.75E-07 |
| 1/64 | 0.82E-03 | 0.22E-03 | 0.54E-04 | 0.13E-04 | 0.34E-05 | 0.84E-06 | 0.21E-06 |
| 1/128 | 0.15E-02 | 0.36E-03 | 0.88E-04 | 0.22E-04 | 0.55E-05 | 0.14E-05 | 0.34E-06 |

Table 4.17: Max. Errors for Example 4.4
Without Using Fitting Factor

| $\varepsilon$ | n = 16 | n = 32 | n = 64 | n = 128 | n = 256 | n = 512 | n = 1024 |
|---|---|---|---|---|---|---|---|
| 1/4 | 0.17E-02 | 0.43E-03 | 0.11E-03 | 0.27E-04 | 0.67E-05 | 0.17E-05 | 0.42E-06 |
| 1/8 | 0.78E-02 | 0.19E-02 | 0.48E-03 | 0.12E-03 | 0.30E-04 | 0.75E-05 | 0.19E-05 |
| 1/16 | 0.35E-01 | 0.79E-02 | 0.19E-02 | 0.48E-03 | 0.12E-03 | 0.30E-04 | 0.75E-05 |
| 1/32 | 0.14E+00 | 0.35E-01 | 0.79E-02 | 0.19E-02 | 0.48E-03 | 0.12E-03 | 0.30E-04 |
| 1/64 | 0.35E+00 | 0.14E+00 | 0.35E-01 | 0.79E-02 | 0.19E-02 | 0.48E-03 | 0.12E-03 |
| 1/128 | 0.60E+00 | 0.35E+00 | 0.14E+00 | 0.35E-01 | 0.79E-02 | 0.19E-02 | 0.48E-03 |
| 1/256 | 0.81E+00 | 0.60E+00 | 0.35E+00 | 0.14E+00 | 0.35E-01 | 0.79E-02 | 0.19E-02 |
| 1/512 | 0.12E+01 | 0.78E+00 | 0.60E+00 | 0.35E+00 | 0.14E+00 | 0.35E-01 | 0.79E-02 |

Table 4.18: Max. Errors for Example 4.4
Using EFSC

| $\varepsilon$ | n = 16 | n = 32 | n = 64 | n = 128 | n = 256 | n = 512 | n = 1024 |
|---|---|---|---|---|---|---|---|
| 1/4 | 0.75E-15 | 0.14E-14 | 0.22E-13 | 0.31E-13 | 0.19E-12 | 0.68E-12 | 0.30E-11 |
| 1/8 | 0.39E-15 | 0.29E-14 | 0.94E-15 | 0.30E-13 | 0.33E-13 | 0.25E-12 | 0.22E-11 |
| 1/16 | 0.28E-15 | 0.67E-15 | 0.51E-14 | 0.20E-14 | 0.79E-13 | 0.54E-13 | 0.20E-12 |
| 1/32 | 0.39E-15 | 0.50E-15 | 0.89E-15 | 0.10E-13 | 0.26E-14 | 0.20E-12 | 0.21E-12 |
| 1/64 | 0.42E-15 | 0.80E-15 | 0.22E-15 | 0.58E-14 | 0.12E-13 | 0.10E-12 | 0.77E-12 |
| 1/128 | 0.11E-15 | 0.11E-14 | 0.22E-14 | 0.17E-15 | 0.11E-13 | 0.23E-13 | 0.21E-12 |
| 1/256 | 0.00E+00 | 0.00E+00 | 0.15E-14 | 0.41E-14 | 0.10E-14 | 0.23E-13 | 0.48E-13 |
| 1/512 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.42E-14 | 0.80E-14 | 0.84E-14 | 0.50E-13 |

Table 4.19: Max. Errors for Example 4.5
Without Using Fitting Factor

| $\varepsilon$ | n = 16 | n = 32 | n = 64 | n = 128 | n = 256 | n = 512 | n = 1024 |
|---|---|---|---|---|---|---|---|
| 1/4 | 0.63E-02 | 0.16E-02 | 0.40E-03 | 0.99E-04 | 0.25E-04 | 0.62E-05 | 0.15E-05 |
| 1/8 | 0.13E-01 | 0.32E-02 | 0.80E-03 | 0.20E-03 | 0.50E-04 | 0.12E-04 | 0.31E-05 |
| 1/16 | 0.42E-01 | 0.94E-02 | 0.23E-02 | 0.57E-03 | 0.14E-03 | 0.36E-04 | 0.89E-05 |
| 1/32 | 0.14E+00 | 0.37E-01 | 0.83E-02 | 0.20E-02 | 0.51E-03 | 0.13E-03 | 0.32E-04 |
| 1/64 | 0.33E+00 | 0.14E+00 | 0.35E-01 | 0.80E-02 | 0.20E-02 | 0.49E-03 | 0.12E-03 |
| 1/128 | 0.54E+00 | 0.34E+00 | 0.13E+00 | 0.35E-01 | 0.79E-02 | 0.19E-02 | 0.48E-03 |
| 1/256 | 0.72E+00 | 0.56E+00 | 0.34E+00 | 0.14E+00 | 0.35E-01 | 0.79E-02 | 0.19E-02 |
| 1/512 | 0.86E+00 | 0.73E+00 | 0.58E+00 | 0.35E+00 | 0.14E+00 | 0.35E-01 | 0.79E-02 |

Table 4.20: Max. Errors for Example 4.5
Using EFSC

| $\varepsilon$ | n = 16 | n = 32 | n = 64 | n = 128 | n = 256 | n = 512 | n = 1024 |
|---|---|---|---|---|---|---|---|
| 1/4 | 0.13E-02 | 0.31E-03 | 0.78E-04 | 0.19E-04 | 0.49E-05 | 0.12E-05 | 0.30E-06 |
| 1/8 | 0.21E-02 | 0.62E-03 | 0.16E-03 | 0.41E-04 | 0.10E-04 | 0.26E-05 | 0.64E-06 |
| 1/16 | 0.64E-02 | 0.20E-02 | 0.54E-03 | 0.14E-03 | 0.35E-04 | 0.86E-05 | 0.22E-05 |
| 1/32 | 0.78E-02 | 0.45E-02 | 0.13E-02 | 0.34E-03 | 0.87E-04 | 0.22E-04 | 0.54E-05 |
| 1/64 | 0.40E-02 | 0.56E-02 | 0.26E-02 | 0.74E-03 | 0.19E-03 | 0.48E-04 | 0.12E-04 |
| 1/128 | 0.45E-02 | 0.27E-02 | 0.32E-02 | 0.14E-02 | 0.39E-03 | 0.10E-03 | 0.25E-04 |
| 1/256 | 0.47E-02 | 0.25E-02 | 0.16E-02 | 0.17E-02 | 0.73E-03 | 0.20E-03 | 0.52E-04 |
| 1/512 | 0.49E-02 | 0.26E-02 | 0.13E-02 | 0.89E-03 | 0.89E-03 | 0.37E-03 | 0.10E-03 |

Table 4.21: Rate of convergence for Example 4.4
Using EFSC
$n = 128, 256, 512, 1024, 2048$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/2 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/4 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/8 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/16 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/32 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/64 | 0.19E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/128 | 0.18E+01 | 0.19E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.19E+01 |
| 1/256 | —— | 0.18E+01 | 0.19E+01 | 0.20E+01 | 0.20E+01 | 0.19E+01 |

Table 4.22: Max. Errors for Example 4.6
$\varepsilon = 10^{-5}$

| $n$ | Veldhuizen ([250]) (UDS) | Veldhuizen ([250]) (MUDS) | Our Results : Using EFSC |
|---|---|---|---|
| 20 | 0.11E-01 | 0.11E-01 | 0.17E-01 |
| 40 | 0.62E-02 | 0.73E-02 | 0.92E-02 |
| 80 | 0.36E-02 | 0.43E-02 | 0.48E-02 |
| 160 | 0.20E-02 | 0.23E-02 | 0.24E-02 |
| 320 | 0.11E-02 | 0.12E-02 | 0.12E-02 |
| 640 | 0.26E-02 | 0.62E-03 | 0.61E-03 |
| 1280 | 0.61E-02 | 0.38E-03 | 0.31E-03 |

Table 4.23: Max. Errors for Example 4.6
$\varepsilon = 10^{-4}$

| | $n = 10$ | $n = 20$ | $n = 40$ | $n = 80$ |
|---|---|---|---|---|
| Veldhuizen ([249]) | 0.16E+00 | 0.79E-01 | 0.39E-01 | 0.20E-01 |
| Ours : Using EFSC | 0.31E-01 | 0.17E-01 | 0.92E-02 | 0.48E-02 |

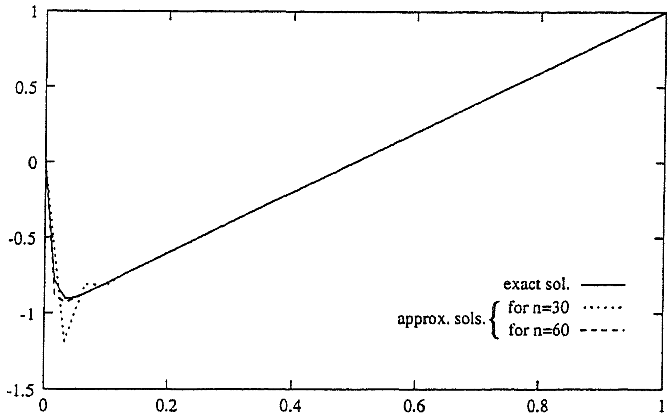


Figure 4.1: Exact and Approx. Solutions of Example 4.4
for $\varepsilon = 0.01$ and $n = 30, 60$ with Uniform Mesh

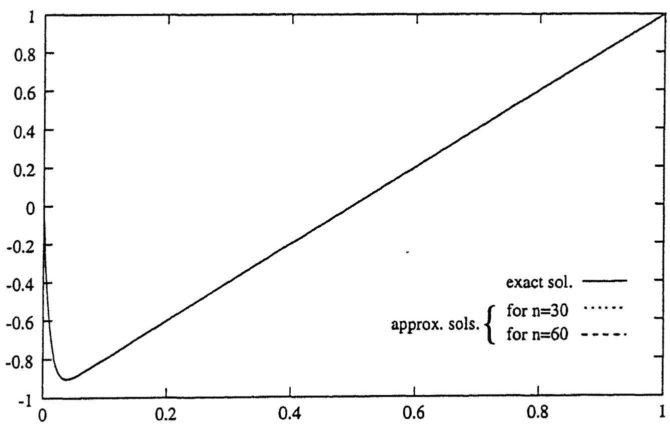


Figure 4.2: Exact and Approx. Solutions of Example 4.4
for $\varepsilon = 0.01$ and $n = 30, 60$ with about 50 % points in B. Layer

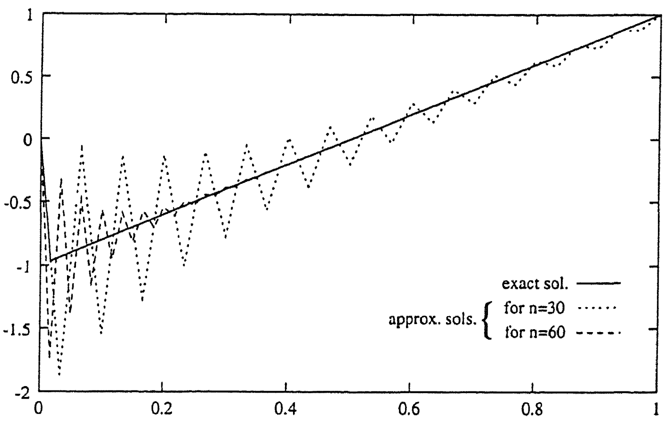


Figure 4.3: Exact and Approx. Solutions of Example 4.4 for $\varepsilon = 0.001$ and $n = 30, 60$ with Uniform Mesh

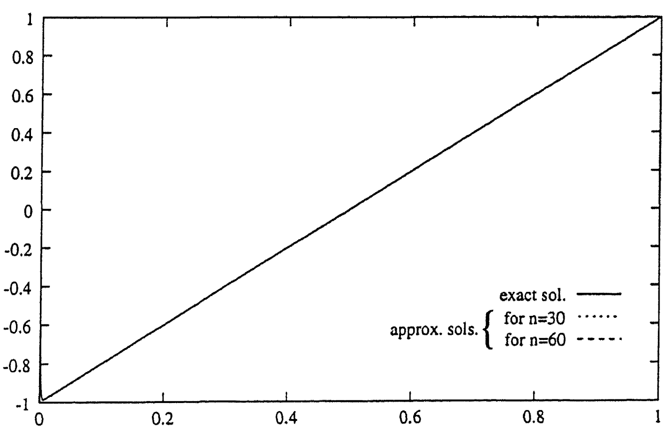


Figure 4.4: Exact and Approx. Solutions of Example 4.4 for $\varepsilon = 0.001$ and $n = 30, 60$ with about 50 % points in B. Layer

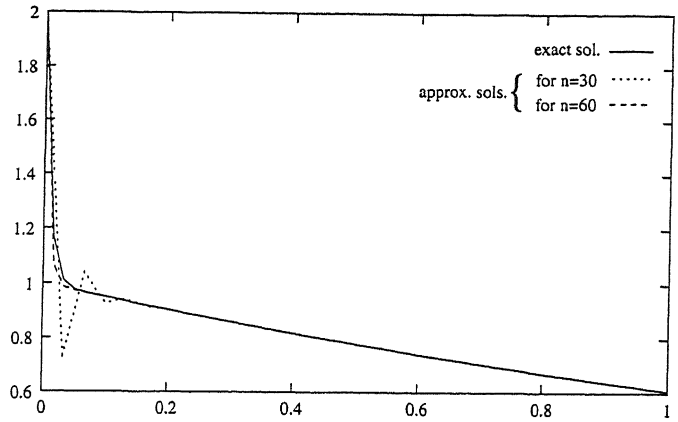


Figure 4.5: Exact and Approx. Solutions of Example 4.5 for $\varepsilon = 0.01$ and $n = 30, 60$ with Uniform Mesh

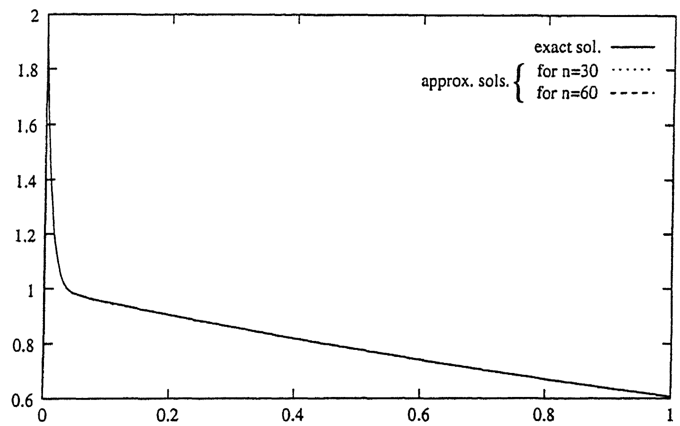


Figure 4.6: Exact and Approx. Solutions of Example 4.5 for $\varepsilon = 0.01$ and $n = 30, 60$ with about 50 % points in B. Layer

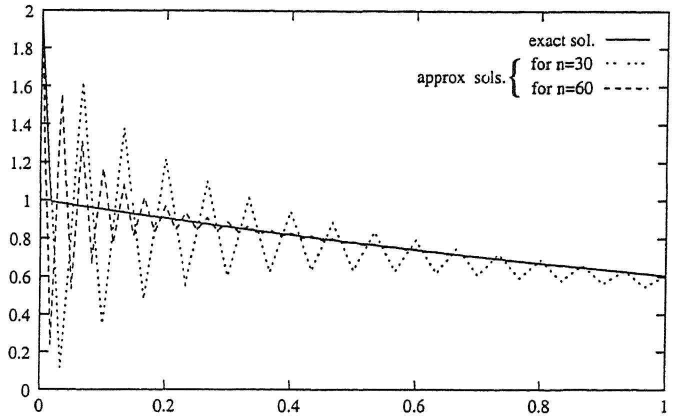


Figure 4.7: Exact and Approx. Solutions of Example 4.5 for $\varepsilon = 0.001$ and $n = 30, 60$ with Uniform Mesh

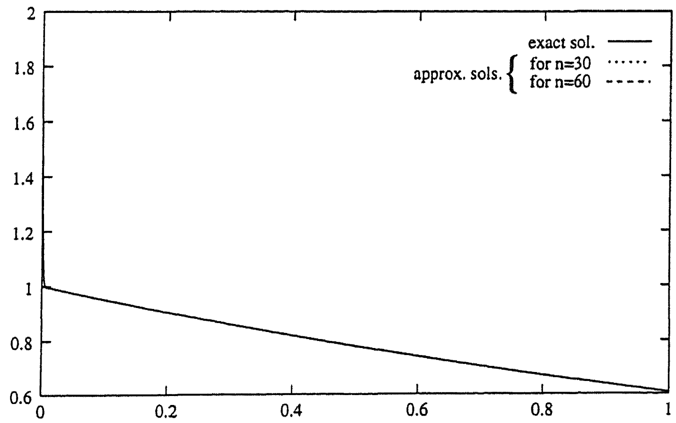


Figure 4.8: Exact and Approx. Solutions of Example 4.5 for $\varepsilon = 0.001$ and $n = 30, 60$ with about 50 % points in B. Layer

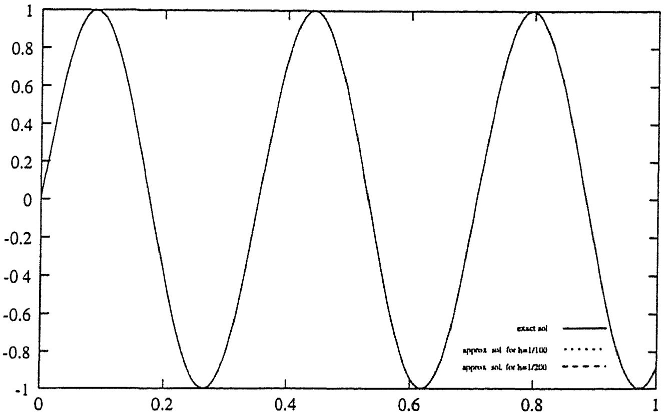


Figure 4.9: Exact and Approx. Solutions of Example 4.2
for $\varepsilon = 1/128$ and $n = 100, 200$, Using Fitting Factor

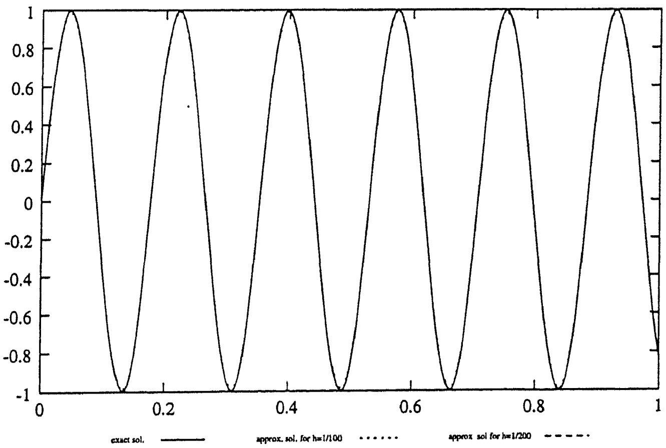


Figure 4.10: Exact and Approx. Solutions of Example 4.2
for $\varepsilon = 1/512$ and $n = 100, 200$, Using Fitting Factor

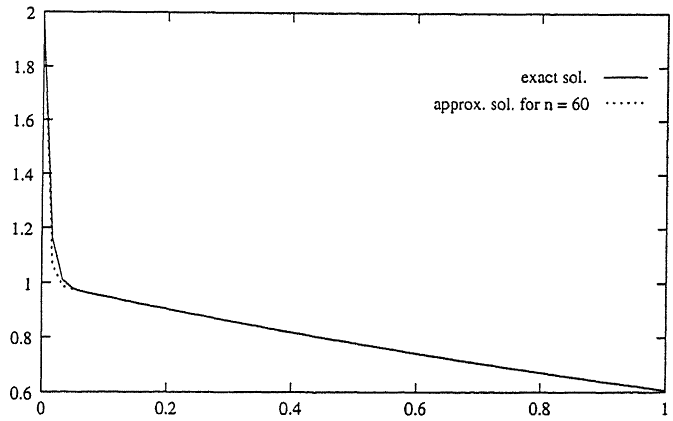


Figure 4.11: Exact and Approx. Solutions of Example 4.5 for $\varepsilon = 0.01$ and $n = 60$, Without Using Fitting Factor

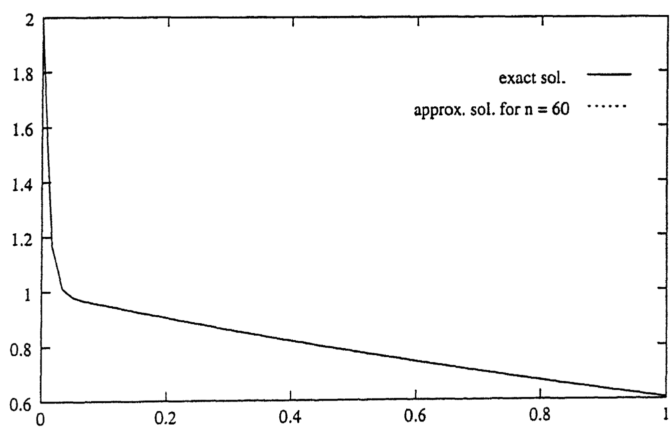


Figure 4.12: Exact and Approx. Solutions of Example 4.5 for $\varepsilon = 0.01$ and $n = 60$, Using Fitting Factor

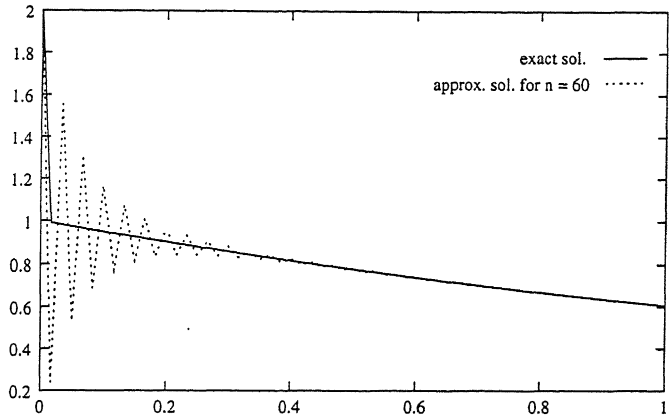


Figure 4.13: Exact and Approx. Solutions of Example 4.5 for $\varepsilon = 0.001$ and $n = 60$, Without Using Fitting Factor

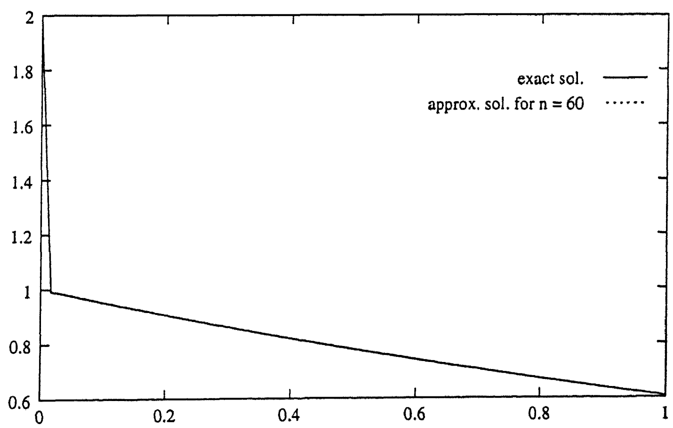


Figure 4.14: Exact and Approx. Solutions of Example 4.5 for $\varepsilon = 0.001$ and $n = 60$, Using Fitting Factor

## 4.6 Discussion

We have described a numerical method for solving singular perturbation problems using VMSC (Variable Mesh Spline in Compression) and EFSC (Exponentially Fitted Spline in Compression). Both of the methods are given for the case when only second and first derivative terms are present. The problem with the second derivative term and the function term is solved by the fitting technique but not by the variable mesh method. The simple reason is that for the case when there is no first derivative term in the most general problem of chapter 2 then the solution is possessing oscillatory behaviour and therefore the mesh refinement (which is generally needed in the boundary layer regions) is not useful whereas the fitting technique still works. This is because of the fact that we assume the existence of the boundary layer for the determination of the fitting factor in that region and then the fitting factor thus obtained is defined for the whole of the interval. This fitting factor takes care of the oscillations. Also in the most general case (considered in chapter 2) we have two consistency conditions involving the parameters $t_j$ and $p_j$ in the scheme. So while using the variable mesh strategy one need to take proper care in the selection of the mesh parameter as it has to satisfy both the consistency conditions simultaneously. As for as the exponential fitting for the most general problem is concerned, we say that if we replace $\varepsilon$ by the fitting factor $\sigma_j$ then the scheme thus obtained will be given by :

$$r_j^- = \left(1 - \frac{t_j^2}{4}\right)\left(\frac{2-p_j}{2+p_j}\right), \quad r_j^+ = \left(1 - \frac{t_{j+1}^2}{4}\right)\left(\frac{2+p_{j+1}}{2-p_{j+1}}\right)$$

$$r_j^c = -2+p_j-p_{j-1}-\frac{1}{4}(t_j^2+t_{j+1}^2),\ q_j^- = \frac{h^2}{2\sigma_j(2+p_j)},\ q_j^+ = \frac{h^2}{2\sigma_{j+1}(2-p_{j+1})},\ q_j^c = q_j^- + q_j^+$$

$$p_j = h\alpha_j/2\sigma_j\ ,\ t_j = [h(\alpha_j^2 - 4\beta_j\sigma_j)^{1/2}]/2\sigma_j$$

Because of these two parameters $p_j$ and $t_j$, the factor $\mu(\rho)$, can not be obtained explicitly and thus the fitting factor has not been determined for this case.

For the numerical solution of Example 4.3, we took $h = 0.1/n$ as the concerned interval for this particular Example is $[0, 0.1]$. In Table 4.22 UDS and MUDS stands for Upstream Difference Scheme and Modified Upwind Difference Scheme, respectively.

In the given mesh selection strategy, the boundary layer width $\delta$ plays an important role. According to Miller et al. [168], if the solution of the homogeneous singular per-

turbation problem involves the functions of the type $exp(-x/\varepsilon)$, then $\delta = O(\varepsilon\, ln\,(1/\varepsilon))$. Moreover our mesh selection procedure needs prior knowledge of $\delta$, $\widetilde{h}_1$ and $K$. Following above we took $\delta = O(\varepsilon\, ln\,(1/\varepsilon))$. The increase in $K$ will lead to more concentration of points near the boundary. Also, for a fixed $K$, the increase in $\widetilde{h}_1$ leads to the same conclusion.

Results are tabulated for both the spline in compression with uniform mesh and with non-uniform mesh and it can be seen from the respective Tables that if we take more mesh points in the boundary layer region then we obtain better results which implies that the use of non-uniform mesh is quite advantageous.

The solution of the equations considered in the first case is having oscillatory behaviour (Bender and Orszag [24] and Flaherty and Mathon [80]). Also $r_j's$ and $q_j's$ in the schemes for case II without using the consistency condition, involves $exp(-\kappa(h,\sigma_j))$ and $exp(+\kappa(h,\sigma_j))$ terms, where $\kappa(h,\sigma_j)$ is a function of $h$ and $\varepsilon$. Therefore in this case the $r_j's$ and $q_j's$ will tend to 0 and(or) $\infty$. Hence the system thus obtained may not be well behaved. To overcome spurious oscillations in the solution of the equation considered in the first case as well as the above mentioned difficulties in the second case, we use the consistency condition.

The problems of the type as considered in this paper have earlier been solved by Stojanovic [222], [223] using exponentially fitted quadratic spline difference schemes, Surla and Jerkovic [236] using exponentially fitted cubic spline collocation method and by Surla and Uzelac [237] using cubic spline difference schemes but they could get only first order of uniform convergence, whereas the present method has second order of uniform convergence. Example 2 has also been solved earlier by Sakai and Usmani [209]. Their method works for more smaller values of $\varepsilon$ but they too did not achieve uniform convergence. However our method works for more smaller values of $\varepsilon$ also but in that case we require $h_j$ also to be very small so as to satisfy the consistency condition.

It can be seen that $r_j's$ and $q_j's$ in the scheme (4.19) without using the consistency condition, involves $exp(-\kappa(h_j,\varepsilon))$ and $exp(+\kappa(h_j,\varepsilon))$ terms, where $\kappa(h_j,\varepsilon)$ is a function of $h_j$ and $\varepsilon$. Therefore in this case the $r_j's$ and $q_j's$ will tend to 0 and(or) $\infty$. Hence the system thus obtained may not be well behaved. To overcome the above mentioned difficulty, we use the consistency condition.

To further corroborate the applicability of the proposed method, graphs have been plotted for examples 4.2,4.4,4.5 for values of $x \in [0, 1]$ versus the computed (termed as approximate) solutions obtained at different values of $x$ for a fixed $\varepsilon$. For each plot we took $n = 30$ and 60. Figures 4.1 and 4.3 are the graphs, using uniform mesh throughout the region, for Example 4.4 for $\varepsilon = 0.01$ and 0.001 respectively, whereas Figures 4.2 and 4.4 are the graphs which are plotted using about 50% mesh points in the boundary layer regions (according to the given mesh selection strategy) for the epsilon values 0.01 and 0.001 respectively and $\delta = O(\varepsilon \, ln(1/\varepsilon))$. Similarly, the graphs for 4.5 are Figures 4.5 and 4.7 using uniform mesh and 4.6 and 4.8 with about 50% mesh points in the boundary layer regions for the similar values of $\varepsilon$ and $\delta$ as were in the case of Figures 4.1-4.4. It can be seen from the Figures 4.1 and 4.5 that the exact and approximate solutions with uniform mesh are identical for most of the range except the boundary layer region where these two solutions deviate from each other. To control these fluctuations, we took more mesh points in the boundary layer region and the resulting behaviour can be seen from the Figures 4.2 and 4.6, respectively, for the corresponding values of $\varepsilon$. As is seen from Figures 4.3 and 4.7 that the approximate solutions fluctuate more from exact solution in the boundary layer region than outside this. This is due to the fact that we took uniform mesh throughout the region and the consistency condition requirement is not satisfied. By using variable mesh these fluctuations are overcomed which can be seen from the Figures 4.4 and 4.8.

It is obvious from Example 4.2 that its exact solution is rapidly oscillatory for small $\varepsilon$. To see the behaviour of our computed solutions with this exact solution, graphs have been plotted for values of $x \in [0, 1]$ versus these two solutions. We took $n = 100$ and 200. Figures 4.9 and 4.10 are respectively the graphs for $\varepsilon = 1/128$ and $\varepsilon = 1/512$. It can be seen that the exact and computed solutions are perfectly identical. Finally, it can be seen that the fitting factor technique is useful for the problems having boundary layer(s) and it can be observed from Figures 4.11, 4.12, 4.13, and 4.14.

Finally, we would like to remark that the two procedures of getting the improved results, viz., the variable mesh method and the exponentially fitted method, may be considered as alternative methods. However, if someone likes to solve the problem on relatively coarser mesh then the exponentially fitted method will be more useful.

# Chapter 5

## SELF-ADJOINT SINGULAR PERTURBATION PROBLEMS

## 5.1 Introduction

We consider the following self-adjoint singularly perturbed two point boundary value problem

$$\left.\begin{array}{rl} Ly \equiv & -\varepsilon\,(a(x)y')' + b(x)y = f(x) \quad \text{on} \quad (0,1) \\ & y(0) = \eta_0, \quad y(1) = \eta_1 \end{array}\right\} \tag{5.1}$$

where, $\eta_0$, $\eta_1$ are given constants and $\varepsilon$ is a small positive parameter. Further, the coefficients $f(x)$, $a(x)$ and $b(x)$ are smooth functions and satisfy

$$a(x) \geq a > 0,\ a'(x) \geq 0,\ b(x) \geq b > 0$$

Under these conditions the operator $L$ admits a maximum principle [196].

Earlier these type of problems have been solved by numerous researchers. Niijima [179] gave uniformly second-order accurate difference schemes whereas Miller [165] gave sufficient conditions for the uniform first-order convergence of a general three-point difference scheme. Boglaev [30] and Schatz and Wahlbin [210] used finite element techniques to solve such problems.

In this chapter we have used the Spline Approximation Methods, to solve the problems of the type (5.1).

We have reduced the original problem (i.e. problem (5.1)) to the normal form. Then we solved the normalized equation by three different methods, referred to as ST, VMCS and EFCS. ST uses the spline in tension discussed in chapter 3, whereas the VMCS and EFCS, respectively, stands for variable mesh cubic spline and exponentially fitted cubic spline.

There are two possibilities to obtain small truncation error inside the boundary layer(s). The first is to choose a fine mesh there whereas the second one is to choose a difference formula reflecting the behaviour of the solution(s) inside the boundary layer(s). Since the spline difference scheme has the same order of precision and the same matrix structure on the uniform and on the non-uniform grid for a fixed $\varepsilon$, we used this property for singularly perturbed problems. This enables us in modifying the distribution of mesh points vis-a-vis to the properties of the exact solution and in turn we devised a variable mesh cubic spline method (VMCS of this chapter). Our EFCS makes use of the above second hypothesis. However, our ST, which is based on tension spline, is given for uniform mesh only.

From the definition of tension spline, it is obvious that the basis involves exponential functions and the boundary layer functions corresponding to the normalized problem are also exponential in nature. Thus no need of exponential fitting for tension splines. Also in the region of nonuniformity, these basis functions as well as the boundary layer functions, both are of same nature (exponential in this case) thus a fine mesh is not going to give much improved results and hence we used cubic spline with variable mesh and with exponential fitting.

Then we applied the three ~~splines~~ to the normal form. By making use of the continuity of the first derivative of the spline function, the resulting spline difference scheme gives a tridiagonal system which can be solved efficiently by the well known algorithms.

If we do not transform problem (5.1) to the normal form and if we discretize (5.1) directly to the normal form then after getting the tridiagonal system one can observe that out of the two diagonal elements, viz., super- and sub-diagonal elements one is exponentially large whereas the other is exponentially small as the parameter $\varepsilon \to 0$ which causes the illconditioning in the system. This happens when $a(x)$ is not a constant. That is why we need to transform problem (5.1) to the normal form. Also in this regard our method is different than that of Surla and Stojanovic [238] where they considered the case $a(x) \equiv 1$ only. So our method is more general than in [238].

In Section 5.2 we gave a brief description of the methods. The derivation of the difference scheme for all the three methods has been given in Section 5.3. The second order accuracy of the method~~s~~ is shown in Section 5.4. In Section 5.5 we have solved

<u>ST</u>:

In this method the approximate solution of the problem (5.5) is sought in the form of the spline function S(x), which on each interval $[x_{j-1}, x_j]$, (denoted by $S_j(x)$) satisfies the following relations :-

(i) the differential equation

$$-\varepsilon S_j''(x) + \widetilde{W}(x) S_j(x) = \widetilde{Z}(x) \tag{5.6}$$

(ii) the interpolating conditions

$$S(x_{j-1}) = \nu_{j-1}, \; S(x_j) = \nu_j \tag{5.7}$$

(iii) the continuity condition

$$S_j'(x_j^+) = S_j'(x_j^-) \tag{5.8}$$

where,

$$x \in [x_{j-1}, x_j], \; x_j = jh, \; j = 1(1)n \text{ and } h = 1/n.$$

and $\widetilde{W}(x)$ and $\widetilde{Z}(x)$ are piecewise polynomial approximations to $W(x)$ and $Z(x)$, respectively.

For $x \in [x_{j-1}, x_j]$, we define $\widetilde{W}(x) = \frac{W_{j-1}+W_j}{2} \equiv \beta_j$ and $\widetilde{Z}(x) = \frac{Z_{j-1}+Z_j}{2} \equiv \gamma_j$

Therefore, in this interval, we need to solve

$$-\varepsilon S_j''(x) + \beta_j S_j(x) = \gamma_j \tag{5.9}$$

Solving equation (5.9) with the help of (5.7), we obtain

$$S_j(x) = \frac{1}{-sinh\ g_j h}\left[P_j\ sinh\ g_j(x_{j-1} - x) + Q_j\ sinh\ g_j(x - x_j)\right] + \frac{\gamma_j}{\beta_j} \tag{5.10}$$

where,

$$P_j = \nu_j - \frac{\gamma_j}{\beta_j}, \; Q_j = \nu_{j-1} - \frac{\gamma_j}{\beta_j} \text{ and } g_j = \sqrt{\frac{\beta_j}{\varepsilon}}$$

Equation (5.10) is known as spline in tension [197]. Using this spline function we will derive the difference scheme in Section 5.3, which will give us the approxiamte solution of $V(x)$. Since $U(x)$ is known, therefore the complete solution will be obtained using (5.3).

VMCS:

In this method the approximate solution of the problem (5.5) is sought in the form of the cubic spline function on each sub interval $[x_{j-1}, x_j]$, denoted by $S_j(x)$ and defined as follows:

Let

$$x_0 = 0 \ , \ x_j = x_0 + \sum_{m=1}^{j} h_m \ , j = 1(1)n, \ h_m = x_m - x_{m-1}, \ x_n = 1$$

For the values $V(x_0)$, $V(x_1)$, $\cdots$ , $V(x_n)$, there exists an interpolating cubic spline with the following properties :

(i) $S_j(x)$ coincides with a polynomial of degree 3 on each interval $[x_{j-1}, x_j], j = 1(1)n$
(ii) $S_j(x) \in C^2[0, 1]$
(iii) $S_j(x_j) = V(x_j), \ j = 0(1)n$

Hence as in [7], the cubic spline can be given as

$$\begin{aligned} S_j(x) \ &= \ \frac{(x_j - x)^3}{6h_j} M_{j-1} + \frac{(x - x_{j-1})^3}{6h_j} M_j + \left(V_{j-1} - \frac{h_j^2 M_{j-1}}{6}\right)\left(\frac{x_j - x}{h_j}\right) \\ &+ \left(V_j - \frac{h_j^2 M_j}{6}\right)\left(\frac{x - x_{j-1}}{h_j}\right) \end{aligned} \tag{5.11}$$

where,

$$x \in [x_{j-1}, x_j] \ , \ h_j = x_j - x_{j-1} \ , \ j = 1(1)n$$

and

$$M_j = S_j''(x_j) \ , \ j = 0(1)n$$

Using this spline function we will derive the difference scheme in Section5.3,which will give us the approximate solution of $V(x)$. Since $U(x)$ is known, therefore the solution to the original problem will be obtained using (5.3).

EFCS:

We define the fitting comparison problem associated with (5.5) by

$$\left.\begin{aligned} -\sigma(x,\varepsilon)V'' \ + \ W(x)V \ &= \ Z(x) \\ V(0) = \alpha_0, \ \ V(1) = \alpha_1 & \end{aligned}\right\} \tag{5.12}$$

where $\sigma(x,\varepsilon)$ is an exponential fitting factor which is to be determined subsequently.

The approximate solution of the problem (5.12) is sought in the form of the cubic spline function, which on each interval $[x_{j-1},x_j]$, denoted by $S_j(x)$ and will be defined as follows:

Let

$$x_0 = 0\ ,\ x_j = x_0 + jh\ ,\ j = 1(1)n\ ,\ h = x_j - x_{j-1}\ ,\ x_n = 1$$

For the values $V(x_0)$, $V(x_1)$, $\cdots$ , $V(x_n)$, there exists an interpolating cubic spline with the following properties:

(i) $S_j(x)$ coincides with a polynomial of degree 3 on each interval $[x_{j-1},x_j], j = 1(1)n$

(ii) $S_j(x) \in C^2[0,1]$

(iii) $S_j(x_j) = V(x_j)$, $j = 0(1)n$

Hence anologous to [7], the cubic spline can be given as :

$$\begin{aligned} S_j(x) &= \frac{(x_j-x)^3}{6h}M_{j-1} + \frac{(x-x_{j-1})^3}{6h}M_j + \left(V_{j-1} - \frac{h^2M_{j-1}}{6}\right)\left(\frac{x_j-x}{h}\right) \\ &\quad + \left(V_j - \frac{h^2M_j}{6}\right)\left(\frac{x-x_{j-1}}{h}\right) \end{aligned} \tag{5.13}$$

where,

$$x \in [x_{j-1},x_j]\ ,\ h = x_j - x_{j-1}\ ,\ j = 1(1)n$$

and

$$M_j = S_j''(x_j)\ ,\ j = 0(1)n$$

Using this spline function we will derive the difference scheme in Section 5.3, which will give us the approximate solution of $V(x)$. Since $U(x)$ is known, therefore the solution to the original problem will be obtained using (5.3).

Remark: Above two definitions of the cubic spline are same. The mere difference is that in the latter case we have $h_j = h = constant$ but to avoid any kind of confussion we have defined them seperately.

## 5.3 Derivation of the scheme

Scheme for ST:

Since $S(x) \in C^2[0,1]$, therefore we have

$$S'_j(x_j) = S'_{j+1}(x_j) \tag{5.14}$$

Differentiating (5.10) with respect to $x$, putting $x = x_j$ and using (5.14), we obtain the difference scheme

$$R\nu_j = QZ_j \ , \ j = 1(1)n-1 \tag{5.15}$$

where,

$$R\nu_j = r_j^- \nu_{j-1} + r_j^c \nu_j + r_j^+ \nu_{j+1}$$

$$QZ_j = q_j^- Z_{j-1} + q_j^c Z_j + q_j^+ Z_{j+1}$$

$$\nu_0 = \alpha_0 \ , \ \nu_n = \alpha_1$$

$$r_j^- = \frac{p_j}{sinh\ p_j}, \ r_j^+ = \frac{p_{j+1}}{sinh\ p_{j+1}}, \ r_j^c = -(p_j\ coth\ p_j + p_{j+1}\ coth\ p_{j+1})$$

$$q_j^- = -\frac{p_j}{2\beta_j}\left(coth\ p_j - \frac{1}{sinh\ p_j}\right), \ q_j^+ = -\frac{p_{j+1}}{2\beta_{j+1}}\left(coth\ p_{j+1} - \frac{1}{sinh\ p_{j+1}}\right)$$

$$q_j^c = q_j^- + q_j^+$$

$$p_j = h\sqrt{\frac{\beta_j}{\varepsilon}}, \ \beta_j = \frac{W_{j-1} + W_j}{2} \tag{5.16}$$

Remark:- The choice of approximation to $\widetilde{W}(x)$ and $\widetilde{Z}(x)$ determines the particular scheme.

If for $x \in [x_{j-1}, x_j]$,

$$\widetilde{W}(x) = \frac{W_{j-1} + W_j}{2} \equiv \beta_j \quad \text{and} \quad \widetilde{Z}(x) = \frac{x - x_{j-1}}{h} Z_j + \frac{x_j - x}{h} Z_{j-1}$$

and for $x \in [x_j, x_{j+1}]$,

$$\widetilde{W}(x) = \frac{W_j + W_{j+1}}{2} \equiv \beta_{j+1} \quad \text{and} \quad \widetilde{Z}(x) = \frac{x - x_j}{h} Z_{j+1} + \frac{x_{j+1} - x}{h} Z_j$$

Then

$$r_j^- = \frac{p_j}{sinh\ p_j}, \ r_j^+ = \frac{p_{j+1}}{sinh\ p_{j+1}}, \ r_j^c = -(p_j\ coth\ p_j + p_{j+1}\ coth\ p_{j+1})$$

$$q_j^- = -\frac{1}{\beta_j}\left(1 - \frac{p_j}{sinh\ p_j}\right), \ q_j^+ = -\frac{1}{\beta_{j+1}}\left(1 - \frac{p_{j+1}}{sinh\ p_{j+1}}\right)$$

$$q_j^c = -\frac{1}{\beta_j}(-1 + p_j\, coth\, p_j) - \frac{1}{\beta_{j+1}}(-1 + p_{j+1}\, coth\, p_{j+1})$$

$$p_j = h\sqrt{\frac{\beta_j}{\varepsilon}} \tag{5.17}$$

If for $x \in [x_{j-1}, x_j]$,

$$\widetilde{W}(x) = W_{j-1} \quad \text{and} \quad \widetilde{Z}(x) = \frac{x - x_{j-1}}{h} Z_j + \frac{x_j - x}{h} Z_{j-1}$$

and for $x \in [x_j, x_{j+1}]$,

$$\widetilde{W}(x) = W_{j+1} \quad \text{and} \quad \widetilde{Z}(x) = \frac{x - x_j}{h} Z_{j+1} + \frac{x_{j+1} - x}{h} Z_j$$

Then

$$r_j^- = \frac{p_{j-1}}{sinh\, p_{j-1}}, \quad r_j^+ = \frac{p_{j+1}}{sinh\, p_{j+1}}, \quad r_j^c = -(p_{j-1}\, coth\, p_{j-1} + p_{j+1}\, coth\, p_{j+1})$$

$$q_j^- = -\frac{1}{W_{j-1}}\left(1 - \frac{p_{j-1}}{sinh\, p_{j-1}}\right), \quad q_j^+ = -\frac{1}{W_{j+1}}\left(1 - \frac{p_{j+1}}{sinh\, p_{j+1}}\right)$$

$$q_j^c = -\frac{1}{W_{j-1}}(-1 + p_{j-1}\, coth\, p_{j-1}) - \frac{1}{W_{j+1}}(-1 + p_{j+1}\, coth\, p_{j+1})$$

$$p_{j-1} = h\sqrt{\frac{W_{j-1}}{\varepsilon}} \quad \text{and} \quad p_{j+1} = h\sqrt{\frac{W_{j+1}}{\varepsilon}} \tag{5.18}$$

If for $x \in [x_{j-1}, x_j]$,

$$\widetilde{W}(x) = W_j \quad \text{and} \quad \widetilde{Z}(x) = \frac{x - x_{j-1}}{h} Z_j + \frac{x_j - x}{h} Z_{j-1}$$

and for $x \in [x_j, x_{j+1}]$,

$$\widetilde{W}(x) = W_j \quad \text{and} \quad \widetilde{Z}(x) = \frac{x - x_j}{h} Z_{j+1} + \frac{x_{j+1} - x}{h} Z_j$$

Then

$$r_j^- = r_j^+ = \frac{p_j}{sinh\, p_j}, \quad r_j^c = -2(p_j\, coth\, p_j)$$

$$q_j^- = q_j^+ = -\frac{1}{W_j}\left(1 - \frac{p_j}{sinh\, p_j}\right)$$

$$q_j^c = -\frac{2}{W_j}(-1 + p_j\, coth\, p_j)$$

$$p_j = h\sqrt{\frac{W_j}{\varepsilon}} \tag{5.19}$$

However we have used (5.16) for error analysis.

Scheme for VMCS:

Differentiating (5.11) and denoting the approximate solution to $V(x)$ by $\nu(x)$, we get

$$S_j'(x) = -\frac{(x_j - x)^2}{2h_j}M_{j-1} + \frac{(x - x_{j-1})^2}{2h_j}M_j + \left(\frac{\nu_j - \nu_{j-1}}{h_j}\right) - \left(\frac{M_j - M_{j-1}}{6}\right)h_j \tag{5.20}$$

and

$$S_{j+1}'(x) = -\frac{(x_{j+1} - x)^2}{2h_{j+1}}M_j + \frac{(x - x_j)^2}{2h_{j+1}}M_{j+1} + \left(\frac{\nu_{j+1} - \nu_j}{h_{j+1}}\right) - \left(\frac{M_{j+1} - M_j}{6}\right)h_{j+1} \tag{5.21}$$

Also since $S_j(x) \in C^2[0,1]$, therefore we have

$$S_j'(x_j) \;=\; S_{j+1}'(x_j) \tag{5.22}$$

Using (5.20), (5.21) and (5.22), we obtain

$$\frac{h_j}{6}M_{j-1} + \frac{h_j + h_{j+1}}{6}M_j + \frac{h_{j+1}}{6}M_{j+1} = \frac{\nu_{j+1} - \nu_j}{h_{j+1}} - \frac{\nu_j - \nu_{j-1}}{h_j} \tag{5.23}$$

Putting $M_j = (\nu_j W_j - Z_j)/\varepsilon$ (from (5.5)) into (5.23) and collecting the coefficients of $\nu_j's$ and $Z_j's$, we have

$$\left(\frac{\varepsilon}{h_j} - \frac{h_j W_{j-1}}{6}\right)\nu_{j-1} - \left[\left(\frac{h_j + h_{j+1}}{3}\right)W_j + \left(\frac{h_j + h_{j+1}}{h_j h_{j+1}}\right)\varepsilon\right]\nu_j$$
$$+\left(\frac{\varepsilon}{h_{j+1}} - \frac{h_{j+1}W_{j+1}}{6}\right)\nu_{j+1} = -\frac{h_j}{6}Z_{j-1} - \frac{h_j + h_{j+1}}{3}Z_j - \frac{h_{j+1}}{6}Z_{j+1}$$

Thus we have obtained the difference scheme

$$R\nu_j \;=\; QZ_j \;,\; j = 1(1)n-1 \tag{5.24}$$

where,

$$R\nu_j \;=\; r_j^- \nu_{j-1} \;+\; r_j^c \nu_j \;+\; r_j^+ \nu_{j+1}$$

$$QZ_j \;=\; q_j^- Z_{j-1} \;+\; q_j^c Z_j \;+\; q_j^+ Z_{j+1}$$

$$\nu_0 \;=\; \alpha_0 \;,\; \nu_n \;=\; \alpha_1$$

$$\left.\begin{aligned} r_j^- &= \tfrac{\varepsilon}{h_j} - \tfrac{h_j W_{j-1}}{6}, \quad r_j^+ = \tfrac{\varepsilon}{h_{j+1}} - \tfrac{h_{j+1} W_{j+1}}{6} \\ r_j^c &= -\left[\left(\tfrac{h_j+h_{j+1}}{3}\right) W_j + \left(\tfrac{h_j+h_{j+1}}{h_j h_{j+1}}\right)\varepsilon\right] \\ q_j^- &= -\tfrac{h_j}{6}, \quad q_j^+ = -\tfrac{h_{j+1}}{6}, \quad q_j^c = -\tfrac{h_j+h_{j+1}}{3} \end{aligned}\right\} \tag{5.25}$$

Mesh Selection Strategy for VMCS:

We form the non-uniform grid in such a way that more points are generated in the boundary layers than outside these regions.

On the interval $[0, \delta]$, where $\delta$ denotes the boundary layer width, the grid is non-uniform and is defined as follows:

Let $n_1 < n$ be the number of mesh points in the boundary layer region $[0, \delta]$ and let the positive constants $\widetilde{h}_1$ and $K$ be known. Then we generate the mesh as:
$\widetilde{h}_j = \widetilde{h}_{j-1} + K(\widetilde{h}_{j-1}/\varepsilon)\min(\widetilde{h}_{j-1}^2, \varepsilon)$ , $j = 2(1)n_1$.
Now, let

$$\widetilde{q} = \sum_{j=1}^{n_1} \widetilde{h}_j$$

$$q = \tfrac{\delta}{\widetilde{q}}$$

and define

$$x_0 = 0$$

$$h_j = q\widetilde{h}_j \ , \ j = 1(1)n_1$$

$$x_j = x_{j-1} + h_j \ , \ j = 1(1)n_1$$

On the interval $[\delta, 1-\delta]$, the grid is uniform and is defined as :

$h_j = \frac{1-2\delta}{n_2}$ , $j = n_1+1, n_1+n_2$ , where $n_2 = n - 2n_1$

$x_j = x_{j-1} + h_j \ , \ j = n_1 + 1, n_1 + n_2$

On the interval $[1-\delta, 1]$, the grid is taken as the mirror image of the grid on $[0, \delta]$.

Scheme for EFCS:

Differentiating (5.13) and denoting the approximate solution to $V(x)$ by $\nu(x)$, we get

$$S_j'(x) = -\frac{(x_j - x)^2}{2h}M_{j-1} + \frac{(x - x_{j-1})^2}{2h}M_j + \left(\frac{\nu_j - \nu_{j-1}}{h}\right) - \left(\frac{M_j - M_{j-1}}{6}\right)h \quad (5.26)$$

Since $S_j(x) \in C^2[0,1]$, therefore we have

$$S_j'(x_j) \ = \ S_{j+1}'(x_j) \quad (5.27)$$

Differentiating (5.26) and (5.27), we obtain the difference scheme

$$R\nu_j \ = \ QZ_j \ \ , \ \ j = 1(1)n-1 \quad (5.28)$$

where,

$$R\nu_j \ = \ r_j^- \nu_{j-1} \ + \ r_j^c \nu_j \ + \ r_j^+ \nu_{j+1}$$

$$QZ_j \ = \ q_j^- Z_{j-1} \ + \ q_j^c Z_j \ + \ q_j^+ Z_{j+1}$$

$$\nu_0 \ = \ \alpha_0 \ \ , \ \ \nu_n \ = \ \alpha_1$$

$$r_j^- = -1\left(1 - \frac{h^2 w_{j-1}}{6\,\sigma_j^-}\right)\frac{1}{h} \ , \ r_j^+ = -1\left(1 - \frac{h^2 w_{j+1}}{6\,\sigma_j^+}\right)\frac{1}{h} \ , \ r_j^c = 2\left(1 + \frac{h^2 w_j}{3\,\sigma_j^c}\right)\frac{1}{h}$$

$$q_j^- = \frac{h}{6\,\sigma_j^-}, \ q_j^+ = \frac{h}{6\,\sigma_j^+}, \ q_j^c = \frac{2h}{3\sigma_j^c} \quad (5.29)$$

where $\sigma_j^- = \sigma_{j-1}$, $\sigma_j^+ = \sigma_{j+1}$ and $\sigma_j^c = \sigma_j$, $\sigma_j$ is to be determined.

Remark:- The scheme without using fitting factor will be given by:

$$r_j^- = -1\left(1 - \frac{h^2 w_{j-1}}{6\,\varepsilon}\right)\frac{1}{h} \ , \ r_j^+ = -1\left(1 - \frac{h^2 w_{j+1}}{6\,\varepsilon}\right)\frac{1}{h} \ , \ r_j^c = 2\left(1 + \frac{h^2 w_j}{3\,\varepsilon}\right)\frac{1}{h}$$

$$q_j^- = \frac{h}{6\,\varepsilon}, \ q_j^+ = \frac{h}{6\,\varepsilon}, \ q_j^c = \frac{2h}{3\varepsilon} \quad (5.30)$$

Determination of the Fitting factor for EFCS

In order to get a suitable fitting factor $\sigma(x, \varepsilon)$, we shall use the following Lemma:-

**Lemma 5.1** [66] *Let $V(x) \in C^4[0,1]$. Let $W'(0) = W'(1) = 0$. Then the solution of the problem (5.5) has the form :-*

$$V(x) = d(x) + e(x) + g(x)$$

*where,*

$$d(x) = q_0 \, exp\left[-x\left\{\frac{W(0)}{\varepsilon}\right\}^{\frac{1}{2}}\right], \quad e(x) = q_1 \, exp\left[-(1-x)\left\{\frac{W(1)}{\varepsilon}\right\}^{\frac{1}{2}}\right]$$

*$q_0$ and $q_1$ are bounded functions of $\varepsilon$ independent of $x$ and*

$$|g^{(k)}(x)| \leq N\left(1 + (\varepsilon)^{1-\frac{k}{2}}\right) \quad , \quad k = 0(1)4$$

*$N$ is a constant independent of $\varepsilon$.*

The matrix of the system (5.28) is inverse monotone if $\frac{h^2 W_i}{6\sigma_i} \leq 1$, $i = j, j \pm 1$. Thus we take a fitting factor in the following way :

$$\sigma_j^- = \frac{h^2 W_{j-1}}{6}\mu(\rho) \, , \, \sigma_j^+ = \frac{h^2 W_{j+1}}{6}\mu(\rho) \, , \, \sigma_j^c = \frac{h^2 W_j}{6}\mu(\rho)$$

where $\mu(\rho)$ will be determined.

We require that the truncation error for the boundary layer functions should be equal to zero when $W(x) = W =$ constant.

From the condition $Rd_j = 0$ for $W(x) = W =$ constant, we have

$$\mu(\rho) = 1 + \frac{3}{2\,sinh^2\left(\frac{\rho h}{2}\right)}$$

The condition $Re_j = 0$ for $W(x) = W =$ constant, will give the same $\mu(\rho)$. Therefore we define

$$\mu(\rho) = 1 + \frac{3}{2\,sinh^2\left(\frac{\rho h}{2}\right)} \, , \quad \text{when } W(x) = W = \text{constant}$$

and

$$\mu(\rho_j) = 1 + \frac{3}{2\,sinh^2\left(\frac{\rho_j h}{2}\right)} \, , \quad \text{when } W(x) \neq \text{constant}$$

Hence the variable fitting factor $\sigma_j$ is defined as :

$$\sigma_j = \frac{h^2 W_j}{6}\mu(\rho_j) \tag{5.31}$$

## 5.4 Proof of the uniform convergence

Throughout the chapter $M$ will denote a positive constant which may take different values in different equations (inequalities) but that are always independent of $h$ and $\varepsilon$.

Proof of the uniform convergence for ST:

The scheme (5.15), (5.16) can be written in the matrix form:

$$A\nu = Z$$

where, A is a matrix of the system (5.15), $\nu$ and $Z$ are corresponding vectors.

Now, the local truncation $\tau_j(\phi)$ of the scheme (5.15), is defined by

$$\tau_j(\phi) = R\phi_j - Q(L\phi)_j$$

where, $\phi(x)$ is an arbitrary sufficiently smooth function.

Therefore,

$$\begin{aligned} \tau_j(V) &= RV_j - Q(LV)_j \\ &= R(V_j - \nu_j) \\ \Rightarrow \quad R(V_j - \nu_j) &= \tau_j(V) \\ \Rightarrow \max_j |V_j - \nu_j| &\leq \|A^{-1}\| \max_j |\tau_j(V)| \end{aligned} \tag{5.32}$$

In order to estimate the values $|V_j - \nu_j|$, we will estimate the truncation error $\tau_j(V)$ and the norm of the matrix $A^{-1}$.

To use the properties of the exact solution of the problem, we use Lemma 5.1.

Estimation of truncation error and the norm of $A^{-1}$:-

From Lemma 5.1, we have

$$\tau_j(V) = \tau_j(d) + \tau_j(e) + \tau_j(g)$$

We will estimate separately the parts of $\tau_j(V)$.

First we consider the case in which $h^2 \leq \varepsilon$:

We will start with $d(x)$:-

$$Rd_j = r_j^- d_{j-1} + r_j^c d_j + r_j^+ d_{j+1}$$

Expanding $d_{j-1}$ and $d_{j+1}$ in terms of $d_j$, we obtain

$$Rd_j = d_j \left[ r_j^- exp(h\sqrt{W_0/\varepsilon}) + r_j^c + r_j^+ exp(-h\sqrt{W_0/\varepsilon}) \right]$$

Expanding exponentials and using (5.16) with

$$\frac{x}{sinh\ x} = 1 - \frac{x^2}{6} + O(x^4)$$

and

$$x\ coth\ x = 1 + \frac{x^2}{3} + O(x^4)$$

and the fact that $d_j$ involves $q_0$ which is a bounded function of $\varepsilon$ independent of $x$, we obtain

$$Rd_j = \frac{h^2}{\varepsilon}(W_0 - W_j)d_j + O\left(h^4\right) \tag{5.33}$$

and

$$\begin{aligned} Q(Ld)_j &= q_j^- Z_{j-1} + q_j^c Z_j + q_j^+ Z_{j+1} \\ &= q_j^-(-\varepsilon d''_{j-1} + W_{j-1}d_{j-1}) + q_j^c(-\varepsilon d''_j + W_j d_j) \\ &\quad + q_j^+(-\varepsilon d''_{j+1} + W_{j+1}d_{j+1}) \end{aligned} \tag{5.34}$$

Now from Lemma 5.1, we have

$$d(x) = q_0\ exp\left[-x\left\{\frac{W(0)}{\varepsilon}\right\}^{\frac{1}{2}}\right]$$

$$\Rightarrow\ d_{j-1} = d_j\ exp\left(h\sqrt{\frac{W_0}{\varepsilon}}\right)\ ,\ d_{j+1} = d_j\ exp\left(-h\sqrt{\frac{W_0}{\varepsilon}}\right)$$

$$d''_{j-1} = \left(\frac{W_0}{\varepsilon}\right) d_{j-1}\ ,\ d''_j = \left(\frac{W_0}{\varepsilon}\right) d_j \ \text{ and }\ d''_{j+1} = \left(\frac{W_0}{\varepsilon}\right) d_{j+1}$$

Putting all these expressions into (5.34), we obtain

$$Q(Ld)_j = \frac{h^2}{\varepsilon}(W_0 - W_j)d_j + O\left(h^4\right) \tag{5.35}$$

From equations (5.33) and (5.35), we have

$$|\tau_j(d)| = |Rd_j - Q(Ld)_j| \leq Mh^4 \tag{5.36}$$

Similarly,

$$\begin{aligned} Re_j &= \frac{h^2}{\varepsilon}(W_1 - W_j)e_j + O\left(h^4\right) && (5.37) \\ Q(Le)_j &= \frac{h^2}{\varepsilon}(W_1 - W_j)e_j + O\left(h^4\right) && (5.38) \\ \Rightarrow \quad |\tau_j(e)| &\leq Mh^4 && (5.39) \end{aligned}$$

Now

$$\tau_j(g) = Rg_j - Q(Lg)_j \tag{5.40}$$

where

$$\begin{aligned} Rg_j &= r_j^- g_{j-1} + r_j^c g_j + r_j^+ g_{j+1} \\ &= (r_j^- + r_j^c + r_j^+)g_j + (r_j^+ - r_j^-)hg_j' + (r_j^+ + r_j^-)\frac{h^2}{2!}g_j'' + \cdots \end{aligned}$$

and

$$\begin{aligned} Q(Lg)_j &= q_j^-(-\varepsilon g_{j-1}'' + W_{j-1}g_{j-1}) + q_j^c(-\varepsilon g_j'' + W_j g_j) + q_j^+(-\varepsilon g_{j+1}'' + W_{j+1}g_{j+1}) \\ &= \left[q_j^- W_{j-1} + q_j^c W_j + q_j^+ W_{j+1}\right] g_j + \left[h\left(-q_j^- W_{j-1} + q_j^+ W_{j+1}\right)\right] g_j' \\ &\quad + \left[q_j^-\left(-\varepsilon + \frac{h^2}{2}W_{j-1}\right) - q_j^c\varepsilon + q_j^+\left(-\varepsilon + \frac{h^2}{2}W_{j+1}\right)\right] g_j'' + \cdots \end{aligned}$$

Therefore from (5.40), we have

$$\tau_j(g) = T_0 g_j + T_1 g_j' + \text{ Remainder terms}$$

where

$$T_0 = (r_j^- + r_j^c + r_j^+) - (q_j^- W_{j-1} + q_j^c W_j + q_j^+ W_{j+1})$$
$$T_1 = (r_j^+ - r_j^-)h - (q_j^+ W_{j+1} - q_j^- W_{j-1})h$$

Using (5.16) we see that $T_0 = 0$ and $|T_1| \leq Mh^4/\varepsilon$, therefore

$$\begin{aligned} |T_1 g_j'| &\leq (Mh^4/\varepsilon)|g_j'| \\ &\leq Mh^4/\varepsilon \quad \text{(using Lemma 5.1)} \\ \Rightarrow \quad |\tau_j(g)| &\leq Mh^4/\varepsilon \end{aligned} \tag{5.41}$$

From (5.36), (5.39) and (5.41), we have

$$|\tau_j(V)| \leq Mh^4/\varepsilon \tag{5.42}$$

Now we consider the case in which $h^2 \geq \varepsilon$:

We introduce the notations $r_j^- = r_j^-(p_{j-1})$, $r_j^+ = r_j^+(p_{j+1})$ and $r_j^c = r_j^c(p_j)$. Putting $p_{j-1} = p_{j+1} = p_j = p_0$ in $Rd_j$, we obtain $Rd_j = 0$. Denoting this expression by $\widetilde{R}d_j$. Thus

$$\begin{aligned} Rd_j &= Rd_j - \widetilde{R}d_j \\ &= \left[r_j^-(p_{j-1}) - r_j^-(p_0)\right] d_{j-1} + \left[r_j^c(p_j) - r_j^c(p_0)\right] d_j \\ &\quad + \left[r_j^+(p_{j+1}) - r_j^+(p_0)\right] d_{j+1} \end{aligned}$$

Since

$$|\beta_{j-1} - \beta_0| \leq Mx_{j-1}^2 \;, \quad |\beta_j - \beta_0| \leq Mx_j^2 \;, \quad |\beta_{j+1} - \beta_0| \leq Mx_{j+1}^2$$

Therefore

$$\left|r_j^-(p_{j-1}) - r_j^-(p_0)\right| \leq Mx_{j-1}^2 h^2/\varepsilon \;, \quad \left|r_j^+(p_{j+1}) - r_j^+(p_0)\right| \leq Mx_{j+1}^2 h^2/\varepsilon$$

$$\left|r_j^c(p_j) - r_j^c(p_0)\right| \leq M\left(x_j^2 + x_{j+1}^2\right) h^2/\varepsilon$$

Therefore

$$|Rd_j| \leq M\frac{h^2}{\varepsilon}\left[x_{j-1}^2 d_{j-1} + \left(x_j^2 + x_{j+1}\right) d_j + x_{j+1}^2 d_{j+1}\right]$$

Now using the fact that (see, e.g., Doolan et al. [66])

$$x\, exp\left(-\frac{cx}{\varepsilon}\right) \leq M\left(\frac{\varepsilon}{c}\right) exp\left(-\frac{cx}{2\varepsilon}\right)$$

and $d_{j's}$ involve $q_0$ which is a bounded function of $\varepsilon$, we obtain

$$x_{j-1}^2 d_{j-1} \leq Mh^2\varepsilon \;, \quad x_j^2 d_j \leq Mh^2\varepsilon \;, \quad x_{j+1}^2 d_{j+1} \leq Mh^2\varepsilon \text{ and } x_{j+1}^2 d_j \leq Mh^2\varepsilon$$

Hence

$$|Rd_j| \leq Mh^4$$

Now

$$Q(Ld)_j = q_j^- d_{j-1}(W_{j-1} - W_0) + q_j^c d_j(W_j - W_0) + q_j^+ d_{j+1}(W_{j+1} - W_0)$$

But

$$\left|q^{\pm c}\right| \leq Mh^2/\varepsilon \text{ and } |W_{j-1} - W_0| \leq Mx_{j-1}^2 \text{ , etc.}$$

Therefore

$$|Q(Ld)_j| \leq M\frac{h^2}{\varepsilon}\left[x_{j-1}^2 d_{j-1} + x_j^2 d_j + x_{j+1}^2 d_{j+1}\right]$$

Hence

$$|Q(Ld)_j| \leq Mh^4$$

Therefore

$$|\tau_j(d)| \leq Mh^4 \tag{5.43}$$

Similarly, we obtain

$$|\tau_j(w)| \leq Mh^4 \tag{5.44}$$

For $\tau_j(g)$ we use the form

$$\tau_j(g) = \left(r_j^+ - r_j^-\right) hg'(\xi_1) + \left(q_j^+ W_{j+1} - q_j^- W_{j-1}\right) hg'(\xi_2) \quad : \; x_{j-1} < \xi_i < x_{j+1} \text{ , } i = 1, 2$$

Now

$$\left|r_j^+ - r_j^-\right| \leq Mh^3/\varepsilon \text{ and } \left|q_j^+ W_{j+1} - q_j^- W_{j-1}\right| \leq Mh^3/\varepsilon$$

Therefore using Lemma 5.1, we obtain

$$|\tau_j(g)| \leq Mh^4/\varepsilon \tag{5.45}$$

From (5.43), (5.44) and (5.45), we have

$$|\tau_j(V)| \leq Mh^4/\varepsilon \tag{5.46}$$

Estimate of $||A^{-1}||$ :-

Since $r_j^c < 0$ and $r_j^\pm > 0$

therefore,

$$||A^{-1}|| \leq \max_j |r_j^- + r_j^c + r_j^+|^{-1} \qquad \text{(Varah [248])}$$

Now for the case $h^2 \leq \varepsilon$, we have

$$\begin{aligned}
|r_j^- + r_j^c + r_j^+| &= \left|\left(\frac{p_j}{sinh\ p_j}\right) - (p_j\ coth\ p_j + p_{j+1}\ coth\ p_{j+1}) + \left(\frac{p_{j+1}}{sinh\ p_{j+1}}\right)\right| \\
&= \left|-\frac{p_j^2}{2} - \frac{p_{j+1}^2}{2}\right| + O\left(\frac{h^4}{\varepsilon^2}\right) \\
&\geq M_1\frac{h^2}{\varepsilon}
\end{aligned}$$

$$\Rightarrow \max_j |r_j^- + r_j^c + r_j^+|^{-1} \leq M\frac{\varepsilon}{h^2} \quad \left(\text{where, } M = \frac{1}{M_1}\right)$$

$$\Rightarrow \qquad ||A^{-1}|| \leq M\frac{\varepsilon}{h^2} \tag{5.47}$$

and for the case $h^2 \geq \varepsilon$, we have

$$\begin{aligned}
|r_j^- + r_j^c + r_j^+| &\geq M_1 p_j\ coth\ p_j \\
&\geq M_2 p_j^2 \\
&\geq M_3 h^2/\varepsilon
\end{aligned}$$

$$\Rightarrow ||A^{-1}|| \leq M\frac{\varepsilon}{h^2} \quad \left(\text{where, } M = \frac{1}{M_3}\right) \tag{5.48}$$

Hence from (5.32), (5.42), (5.46), (5.47) and (5.48), we have the following theorem:

**Theorem 5.1** *Let $W(x)$, $Z(x) \in C^2[0,1]$ and $W(x) \geq W > 0, W'(0) = W'(1) = 0$. Let $\nu_j$, j=0(1)n, be the approximate solution of (5.5), obtained using (5.15), (5.16). Then, there is a constant $M$ independent of $\varepsilon$ and $h$ such that*

$$\max_j |V(x_j) - \nu_j| \leq Mh^2$$

Proof of the uniform convergence for VMCS:

The scheme (5.24), (5.25) can be written in the matrix form:

$$A\nu = Z$$

where, $A$ is a matrix of the system (5.24), $\nu$ and $Z$ are corresponding vectors.

Now, the local truncation error $\tau_j(\phi)$ of the scheme (5.24), is defined by

$$\tau_j(\phi) = R\phi_j - Q(L\phi)_j$$

where, $\phi(x)$ is an arbitrary sufficiently smooth function.

Therefore,

$$\begin{aligned} \tau_j(V) &= RV_j - Q(L\nu)_j \\ &= R(V_j - \nu_j) \\ \Rightarrow \quad R(V_j - \nu_j) &= \tau_j(V) \\ \Rightarrow \max_j |V_j - \nu_j| &\leq \|A^{-1}\| \max_j |\tau_j(V)| \end{aligned} \tag{5.49}$$

In order to estimate the values $|V_j - \nu_j|$, we will estimate the truncation error $\tau_j(V)$ and the norm of the matrix $A^{-1}$.

Again for the properties of the exact solution we use the Lemma 5.1.

Estimation of truncation error and the norm of $A^{-1}$:-

From Lemma 5.1, we have

$$\tau_j(V) = \tau_j(d) + \tau_j(e) + \tau_j(g)$$

We will estimate separately the parts of $\tau_j(V)$.

First we consider the case in which $h_j^2 \leq \varepsilon$:

We will start with $d(x)$ :-

$$Rd_j = r_j^- d_{j-1} + r_j^c d_j + r_j^+ d_{j+1} \tag{5.50}$$

and

$$\begin{aligned} Q(Ld)_j &= q_j^- Z_{j-1} + q_j^c Z_j + q_j^+ Z_{j+1} \\ &= q_j^-(-\varepsilon d''_{j-1} + W_{j-1}d_{j-1}) + q_j^c(-\varepsilon d''_j + W_j d_j) \\ &\quad + q_j^+(-\varepsilon d''_{j+1} + W_{j+1}d_{j+1}) \end{aligned} \tag{5.51}$$

Now from Lemma 5.1, we have

$$d(x) = q_0\, exp\left[-x\left\{\frac{W(0)}{\varepsilon}\right\}^{\frac{1}{2}}\right]$$

$$\Rightarrow \quad d_{j-1} = d_j\, exp\left(h_j\sqrt{\frac{W_0}{\varepsilon}}\right) \quad , \quad d_{j+1} = d_j\, exp\left(-h_{j+1}\sqrt{\frac{W_0}{\varepsilon}}\right)$$

$$d''_{j-1} = \left(\frac{W_0}{\varepsilon}\right) d_j\, exp\left(h_j\sqrt{\frac{W_0}{\varepsilon}}\right) \quad , \quad d''_{j+1} = \left(\frac{W_0}{\varepsilon}\right) d_j\, exp\left(-h_{j+1}\sqrt{\frac{W_0}{\varepsilon}}\right)$$

$$d''_j = \left(\frac{W_0}{\varepsilon}\right) d_j$$

Putting all these expressions into (5.50) and (5.51) and since

$$\tau_j(d) = Rd_j - Q(Ld)_j \tag{5.52}$$

therefore, we get

$$\begin{aligned} \tau_j(d) &= d_j \left[\left(\frac{\varepsilon}{h_j} - \frac{h_j W_0}{6}\right) exp\left(h_j\sqrt{\frac{W_0}{\varepsilon}}\right) + \left(\frac{\varepsilon}{h_{j+1}} - \frac{h_{j+1}W_0}{6}\right) exp\left(-h_{j+1}\sqrt{\frac{W_0}{\varepsilon}}\right)\right. \\ &\quad \left. -\varepsilon\left(\frac{1}{h_{j+1}} + \frac{1}{h_j}\right) - \frac{W_0}{3}(h_{j+1} + h_j)\right] \end{aligned}$$

Expanding exponentials, we get

$$\tau_j(d) = -\frac{w_0^2}{12\varepsilon} d_j(h_j^3 + h_{j+1}^3) + \text{higher order terms}$$

Now according to our mesh selection strategy, we have

$$\max\,(h_j, h_{j+1}) = h_{j+1} \quad \text{in the interval } [0, \delta] \tag{5.53}$$

$$\Rightarrow \quad |\tau_j(d)| \;\leq\; \frac{Mh_{j+1}^3 d_j}{\varepsilon}$$

But, the expression for $d_j$ involves $q_0$ in the numerator and $q_0$ is a bounded function of $\varepsilon$ independent of $x$. Therefore we have

$$|\tau_j(d)| \;\leq\; Mh_{j+1}^3 \tag{5.54}$$

Now

$$e(x) = q_1\, exp\left[-(1-x)\left\{\frac{W(1)}{\varepsilon}\right\}^{\frac{1}{2}}\right]$$

$$\Rightarrow \qquad e_{j-1} = e_j\ exp\left(-h_j\sqrt{\frac{W_1}{\varepsilon}}\right) \quad , \quad e_{j+1} = e_j\ exp\left(h_{j+1}\sqrt{\frac{W_1}{\varepsilon}}\right)$$

$$e''_{j-1} = \left(\frac{W_1}{\varepsilon}\right)\ e_j\ exp\left(-h_j\sqrt{\frac{W_1}{\varepsilon}}\right) \quad , \quad e''_{j+1} = \left(\frac{W_1}{\varepsilon}\right)\ e_j\ exp\left(h_{j+1}\sqrt{\frac{W_1}{\varepsilon}}\right)$$

$$e''_j = \left(\frac{W_1}{\varepsilon}\right)\ e_j$$

and the similar construction as was for $d(x)$, will give us

$$|\tau_j(e)| \ \leq \ Mh_j^3 \tag{5.55}$$

where we have used the fact that in $[1-\delta, 1]$,

$$\max\ (h_j, h_{j+1}) = h_j \tag{5.56}$$

Now

$$\tau_j(g) \ = \ Rg_j - Q(Lg)_j \tag{5.57}$$

and since in $[\delta, 1-\delta]$ the mesh is uniform, therefore in this region we have

$$h_{j+1} = h_j \ , \ \forall\ j \tag{5.58}$$

Therefore, from (5.57), we get

$$\tau_j(g) = \frac{\varepsilon}{h_j}(g_{j-1} - 2g_j + g_{j+1}) - \frac{\varepsilon h_j}{6}(g''_{j-1} + 4g''_j + g''_{j+1}) \tag{5.59}$$

Expanding $g_{j-1}$, $g_{j+1}$ and their derivatives in terms of $g_j$ and its derivatives, we get

$$|\tau_j(g)| \leq M\varepsilon h_j^3 |g_j^{(iv)}|$$

Therefore, using Lemma 5.1, we obtain

$$|\tau_j(g)| \ \leq \ Mh_j^3 \tag{5.60}$$

From (5.54), (5.55) and (5.60), we have

$$|\tau_j(V)| \ \leq \ M(h_{j+1}^3 + h_j^3) \tag{5.61}$$

Now we consider the case in which $h_j^2 \geq \varepsilon$:

We introduce the notations $r_j^- = r_j^-(W_{j-1})$, $r_j^+ = r_j^+(W_{j+1})$ and $r_j^c = r_j^c(W_j)$. Putting $W_{j-1} = W_{j+1} = W_j = W_0$ in $Rd_j$, we obtain $Rd_j = 0$. Denoting this expression by $\widetilde{R}d_j$. Thus

$$\begin{aligned} Rd_j &= Rd_j - \widetilde{R}d_j \\ &= \left[r_j^-(W_{j-1}) - r_j^-(W_0)\right] d_{j-1} + \left[r_j^c(W_j) - r_j^c(W_0)\right] d_j \\ &\quad + \left[r_j^+(W_{j+1}) - r_j^+(W_0)\right] d_{j+1} \end{aligned}$$

Since

$$|W_{j-1} - W_0| \le Mx_{j-1}^2 \ , \ \ |W_j - W_0| \le Mx_j^2 \ , \ \ |W_{j+1} - W_0| \le Mx_{j+1}^2$$

Therefore

$$\left|r_j^-(W_{j-1}) - r_j^-(W_0)\right| \le Mx_{j-1}^2 h_c \ , \ \ \left|r_j^c(W_j) - r_j^c(W_0)\right| \le Mx_j^2 h_c$$

$$\left|r_j^+(W_{j+1}) - r_j^+(W_0)\right| \le Mx_{j+1}^2 h_c$$

where, $h_c = \max h_j =$ constant. Therefore

$$|Rd_j| \le Mh_c \left[x_{j-1}^2 d_{j-1} + x_j^2 d_j + x_{j+1}^2 d_{j+1}\right]$$

Now using the fact that (see, e.g., Doolan et al. [66])

$$x\, exp\left(-\frac{cx}{\varepsilon}\right) \le M\left(\frac{\varepsilon}{c}\right) exp\left(-\frac{cx}{2\varepsilon}\right)$$

and $d_{j's}$ involve $q_0$ which is a bounded function of $\varepsilon$, we obtain

$$x_{j-1}^2 d_{j-1} \le M\varepsilon^2 \ , \ \ x_j^2 d_j \le M\varepsilon^2 \ \text{ and } \ x_{j+1}^2 d_{j+1} \le M\varepsilon^2$$

Hence

$$|Rd_j| \le Mh_c\varepsilon^2$$

Now

$$Q(Ld)_j = q_j^- d_{j-1}(W_{j-1} - W_0) + q_j^c d_j(W_j - W_0) + q_j^+ d_{j+1}(W_{j+1} - W_0)$$

But

$$\left|q^{\pm c}\right| \le Mh_c \ \text{ and } \ |W_{j-1} - W_0| \le Mx_{j-1}^2 \text{ , etc.}$$

Therefore

$$|Q(Ld)_j| \leq Mh_c\left[x_{j-1}^2 d_{j-1} + x_j^2 d_j + x_{j+1}^2 d_{j+1}\right]$$

$$\Rightarrow |Q(Ld)_j| \leq Mh_c\varepsilon^2$$

Hence

$$|\tau_j(d)| \leq Mh_c\varepsilon^2$$

Since $0 < \varepsilon < 1 \Rightarrow \varepsilon^2 < \varepsilon$. Therefore, we have

$$|\tau_j(d)| \leq Mh_c\varepsilon \tag{5.62}$$

Similarly, we obtain

$$|\tau_j(e)| \leq Mh_c\varepsilon \tag{5.63}$$

Since

$$\tau_j(V) = RV_j - Q(LV)_j$$

$$\Rightarrow \tau_j(V) = \frac{\varepsilon}{6}\left[(h_j + h_{j+1})V_j'' - h_j V_{j-1}'' - h_{j+1}V_{j+1}''\right] + \text{Remainder terms}$$

Therefore for $\tau_j(g)$ we use the form

$$\tau_j(g) = -\frac{\varepsilon h_j}{6}g''(\xi_1) + \frac{\varepsilon h_j}{3}g''(\xi_2) - \frac{\varepsilon h_j}{6}g''(\xi_3) \quad : \; x_{j-1} < \xi_i < x_{j+1}\,,\; i = 1,2,3$$

where, $h_{j+1} = h_j \;,\; \forall\; j$ as the mesh is uniform in $[\delta, 1-\delta]$.

Therefore using Lemma 5.1, we obtain

$$|\tau_j(g)| \leq Mh_c\varepsilon \tag{5.64}$$

From (5.62), (5.63) and (5.64), we have

$$|\tau_j(V)| \leq Mh_c\varepsilon \tag{5.65}$$

Estimate of $\|A^{-1}\|$ :-

Following Varah [248], we see that

$$
\begin{aligned}
|-r_j^- + r_j^c - r_j^+| &= \left| -\left(\frac{\varepsilon}{h_j} - \frac{h_j W_{j-1}}{6}\right) + \left(-\frac{h_j + h_{j+1}}{3} W_j - \frac{h_j + h_{j+1}}{h_j h_{j+1}} \varepsilon\right) \right. \\
&\quad \left. - \left(\frac{\varepsilon}{h_{j+1}} - \frac{h_{j+1} W_{j+1}}{6}\right) \right| \\
&\geq M_1 \left(\frac{\varepsilon + h_c^2}{h_c}\right), \quad \text{where, } h_c = \max h_j
\end{aligned}
$$

$$
\Rightarrow |-r_j^- + r_j^c - r_j^+| \geq \begin{cases} M_1 h_c, & h_c^2 \leq \varepsilon \\ M_1 \varepsilon / h_c, & h_c^2 \geq \varepsilon \end{cases}
$$

$$
\Rightarrow \qquad \|A^{-1}\| \leq \begin{cases} M_2 / h_c, & h_c^2 \leq \varepsilon \\ M_2 h_c / \varepsilon; , & h_c^2 \geq \varepsilon \end{cases} \tag{5.66}
$$

where, $M_1$ and $M_2 (= 1/M_1)$ are constants independent of $h$ and $\varepsilon$.

From (5.49), (5.61), (5.65) and (5.66) , we have the following theorem:

**Theorem 5.2** *Let $W(x)$, $Z(x) \in C^2[0,1]$ and $W(x) \geq W > 0, W'(0) = W'(1) = 0$. Let $\nu_j$, j=0(1)n, be the approximate solution of (5.5), obtained using (5.24), (5.25). Then, there is a constant $M$ independent of $\varepsilon$ and $h$ such that*

$$
\max_j |V(x_j) - \nu_j| \leq M h_c^2
$$

where, $h_c = \max h_j =$ constant.

<u>Proof of the uniform convergence for EFCS:</u>

The scheme (5.28), (5.29) can be written in the matrix form :

$$
A\nu = Z
$$

where, $A$ is a matrix of the system (5.28), $\nu$ and $Z$ are corresponding vectors.

Now, the local truncation error $\tau_j(\phi)$ of the scheme (5.28), is defined by

$$
\tau_j(\phi) = R\phi_j - Q(L\phi)_j
$$

where, $\phi(x)$ is an arbitrary sufficiently smooth function.

Therefore,

$$\begin{aligned} \tau_j(V) &= RV_j - Q(L\nu)_j \\ &= R(V_j - \nu_j) \\ \Rightarrow \quad R(V_j - \nu_j) &= \tau_j(V) \\ \Rightarrow \max_j |V_j - \nu_j| &\leq \|A^{-1}\| \max_j |\tau_j(V)| \end{aligned} \tag{5.67}$$

In order to estimate the values $|V_j - \nu_j|$, we will estimate the truncation error $\tau_j(V)$ and the norm of the matrix $A^{-1}$.

From (5.31) it is obvious that

$$0 \leq \sigma_j \leq Mh^2$$

$$\Rightarrow |\sigma_j - \varepsilon| \leq Mh^2 \text{ for } \varepsilon \leq h^2$$

Now for the case $h^2 \leq \varepsilon$, we see that

$$\sigma_j - \varepsilon = \frac{h^2 W_j}{6} + \varepsilon \left[ \frac{\left(\frac{h\rho_j}{2}\right)^2}{sinh^2\left(\frac{h\rho_j}{2}\right)} - 1 \right]$$

$$\Rightarrow |\sigma_j - \varepsilon| \leq Mh^2 \text{ for } h^2 \leq \varepsilon$$

Hence

$$|\sigma_j - \varepsilon| \leq Mh^2 \tag{5.68}$$

i.e. $\sigma_j$ approximates $\varepsilon$ with the error $O(h^2)$.

<u>Estimation of truncation error and the norm of $A^{-1}$</u>:-

From Lemma 5.1, we have

$$\tau_j(V) = \tau_j(d) + \tau_j(e) + \tau_j(g)$$

We will estimate separately the parts of $\tau_j(V)$.

First we consider the case in which $h^2 \leq \varepsilon$:

We will start with $d(x)$ :-

$$Rd_j \;=\; r_j^- d_{j-1} + r_j^c d_j + r_j^+ d_{j+1} \tag{5.69}$$

and

$$\begin{aligned} Q(Ld)_j &= q_j^- Z_{j-1} + q_j^c Z_j + q_j^+ Z_{j+1} \\ &= q_j^-(-\varepsilon d''_{j-1} + W_{j-1} d_{j-1}) + q_j^c(-\varepsilon d''_j + W_j d_j) \\ &\quad + q_j^+(-\varepsilon d''_{j+1} + W_{j+1} d_{j+1}) \end{aligned} \tag{5.70}$$

Now from Lemma 5.1, we have

$$d(x) \;=\; q_0 \; exp\left[ -x \left\{ \frac{W(0)}{\varepsilon} \right\}^{\frac{1}{2}} \right]$$

$$\Rightarrow \qquad d_{j-1} = d_j \; exp\left( h\sqrt{\frac{W_0}{\varepsilon}} \right) \;,\; d_{j+1} = d_j \; exp\left( -h\sqrt{\frac{W_0}{\varepsilon}} \right)$$

$$d''_{j-1} = \left( \frac{W_0}{\varepsilon} \right) d_j \; exp\left( h\sqrt{\frac{W_0}{\varepsilon}} \right) \;,\; d''_{j+1} = \left( \frac{W_0}{\varepsilon} \right) d_j \; exp\left( -h\sqrt{\frac{W_0}{\varepsilon}} \right)$$

$$d''_j = \left( \frac{W_0}{\varepsilon} \right) d_j$$

Putting all these expressions into (5.69) and (5.70) and since

$$\tau_j(d) = Rd_j - Q(Ld)_j \tag{5.71}$$

therefore, we get

$$\tau_j(d) \;=\; d_{j-1}\left( -\frac{1}{h} + \frac{hW_0}{6\sigma_j^-} \right) + d_j\left( \frac{2}{h} + \frac{2hW_0}{3\sigma_j^c} \right) + d_{j+1}\left( -\frac{1}{h} + \frac{hW_0}{6\sigma_j^+} \right)$$

From (5.68), $\sigma_j = \varepsilon + O(h^2)$ and using the above expressions for $d_{j-1}$ and $d_{j+1}$, we have

$$\Rightarrow \qquad |\tau_j(d)| \;\leq\; \frac{Mh^3 d_j}{\varepsilon^2}$$

But the expression for $d(x)$ involves $q_0$ in the numerator which is a bounded function of $\varepsilon$ independent of $x$. Therefore, we get

$$|\tau_j(d)| \;\leq\; \frac{Mh^3}{\varepsilon} \tag{5.72}$$

Now

$$e(x) = q_1\, exp\left[-(1-x)\left\{\frac{W(1)}{\varepsilon}\right\}^{\frac{1}{2}}\right]$$

$$\Rightarrow \qquad e_{j-1} = e_j\, exp\left(-h\sqrt{\frac{W_1}{\varepsilon}}\right) \quad , \quad e_{j+1} = e_j\, exp\left(h\sqrt{\frac{W_1}{\varepsilon}}\right)$$

$$e''_{j-1} = \left(\frac{W_1}{\varepsilon}\right)\, e_j\, exp\left(-h\sqrt{\frac{W_1}{\varepsilon}}\right) \quad , \quad e''_{j+1} = \left(\frac{W_1}{\varepsilon}\right)\, e_j\, exp\left(h\sqrt{\frac{W_1}{\varepsilon}}\right)$$

$$e''_j = \left(\frac{W_1}{\varepsilon}\right)\, e_j$$

and the similar construction as was for $d(x)$, will give us

$$|\tau_j(e)| \leq \frac{Mh^3}{\varepsilon} \tag{5.73}$$

Now

$$\tau_j(g) = Rg_j - Q(Lg)_j \tag{5.74}$$

$$\Rightarrow \tau_j(g) = -\frac{1}{h}\,(g_{j-1} - 2g_j + g_{j+1}) + \frac{\varepsilon h}{6}\left(\frac{g''_{j-1}}{\sigma_j^-} + \frac{4g''_j}{\sigma_j^c} + \frac{g''_{j+1}}{\sigma_j^+}\right) \tag{5.75}$$

Expanding $g_{j-1}$, $g_{j+1}$ and their derivatives in terms of $g_j$ and its derivatives and using (5.68), we get

$$|\tau_j(g)| \leq Mh^3|g_j^{(iv)}|$$

Therefore, using Lemma 5.1, we obtain

$$|\tau_j(g)| \leq \frac{Mh^3}{\varepsilon} \tag{5.76}$$

From (5.72), (5.73) and (5.76), we have

$$|\tau_j(V)| \leq \frac{Mh^3}{\varepsilon} \text{ when } h^2 \leq \varepsilon \tag{5.77}$$

Now we consider the case in which $h^2 \geq \varepsilon$:

We introduce the notations

$$r_j^- = r_j^-(W_{j-1}) \quad , \quad r_j^+ = r_j^+(W_{j+1}) \quad , \quad r_j^c = r_j^c(W_j)$$

and

$$q_j^- = q_j^-(W_{j-1}) \quad , \quad q_j^+ = q_j^+(W_{j+1}) \quad , \quad q_j^c = q_j^c(W_j)$$

Since we have determined $\sigma(x,\varepsilon)$ in such a way that the truncation error for the boundary layer function(s) is equal to zero in the case of $W(x) = W =$ constant. Thus $\tau_j(d) = 0$ when $W(x) = W =$ constant. We shall denote this expression by $\widetilde{\tau}_j(d)$. Therefore

$$\begin{aligned}
\tau_j(d) &= \tau_j(d) - \widetilde{\tau}_j(d) \\
&= \left[\left\{r_j^-(W_{j-1}) - r_j^-(W_0)\right\} - \left\{q_j^-(W_{j-1}) - q_j^-(W_0)\right\}(W_{j-1} - W_0)\right] d_{j-1} \\
&\quad + \left[\left\{r_j^c(W_j) - r_j^c(W_0)\right\} - \left\{q_j^c(W_j) - q_j^c(W_0)\right\}(W_j - W_0)\right] d_j \\
&\quad + \left[\left\{r_j^+(W_{j+1}) - r_j^+(W_0)\right\} - \left\{q_j^+(W_{j+1}) - q_j^+(W_0)\right\}(W_{j+1} - W_0)\right] d_{j+1}
\end{aligned}$$

Using (5.29), (5.68) and since

$$|W_{j-1} - W_0| \le Mx_{j-1}^2 \ , \quad |W_j - W_0| \le Mx_j^2 \ , \quad |W_{j+1} - W_0| \le Mx_{j+1}^2$$

we obtain

$$|\tau_j(d)| \le \frac{M}{h}\left[x_{j-1}^2 d_{j-1} + x_j^2 d_j + x_{j+1}^2 d_{j+1}\right]$$

Now using the fact that (see, e.g., Doolan et al. [66])

$$x\,exp\left(-\frac{cx}{\varepsilon}\right) \le M\left(\frac{\varepsilon}{c}\right)\,exp\left(-\frac{cx}{2\varepsilon}\right)$$

and $d_{j's}$ involve $q_0$ which is a bounded function of $\varepsilon$, we obtain

$$x_{j-1}^2 d_{j-1} \le M\varepsilon^2 \ , \quad x_j^2 d_j \le M\varepsilon^2 \ \text{ and } \ x_{j+1}^2 d_{j+1} \le M\varepsilon^2$$

Hence

$$|\tau_j(d)| \le \frac{M\varepsilon^2}{h}$$

But

$$\varepsilon < 1 \Rightarrow \varepsilon^2 < \varepsilon$$

Therefore

$$|\tau_j(d)| \le \frac{M\varepsilon}{h}$$

$$\Rightarrow |\tau_j(d)| \le Mh \quad (\text{since} \quad \varepsilon \le h^2) \tag{5.78}$$

Similarly, we obtain

$$|\tau_j(e)| \le Mh \tag{5.79}$$

Now for $\tau_j(g)$, we use the form

$$\tau_j(g) = \frac{\varepsilon}{h}\left(g''_{j-1} + 4g''_j + g''_{j+1}\right) + hg''(\xi) \quad : \quad x_{j-1} < \xi < x_{j+1}$$

Therefore using Lemma 5.1, we obtain

$$|\tau_j(g)| \;\leq\; Mh \tag{5.30}$$

From (5.78), (5.79) and (5.80), we have

$$|\tau_j(V)| \leq Mh \tag{5.31}$$

Estimate of $||A^{-1}||$ :-

Following Varah [248], we see that

$$\begin{aligned}
\left|-r_j^- + r_j^c - r_j^+\right| &= \left|\frac{1}{h} - \frac{hW_{j-1}}{6\sigma_j^-} + \frac{2}{h} + \frac{2hW_j}{3\sigma_j^c} + \frac{1}{h} - \frac{hW_{j+1}}{6\sigma_j^+}\right| \\
&= \left|\frac{4}{h} - \frac{h}{6}\left(W_{j-1} - 4W_j + W_{j+1}\right)\frac{1}{\varepsilon + O(h^2)}\right| \qquad \text{(using (5.68))}
\end{aligned}$$

$$\left|-r_j^- + r_j^c - r_j^+\right| \;\geq\; \begin{cases} \frac{M_1 h}{\varepsilon} & , \quad h^2 \leq \varepsilon \\ \frac{M_1}{h} & , \quad h^2 \geq \varepsilon \end{cases}$$

$$\Rightarrow \qquad ||A^{-1}|| \;\leq\; \begin{cases} \frac{M_2 \varepsilon}{h} & , \quad h^2 \leq \varepsilon \\ M_2 h & , \quad h^2 \geq \varepsilon \end{cases} \tag{5.82}$$

where, $M_1$ and $M_2(= 1/M_1)$ are constants independent of $h$ and $\varepsilon$.

From (5.67), (5.77), (5.81) and (5.82), we have the following theorem:

**Theorem 5.3** *Let $W(x)$, $Z(x) \in C^2[0,1]$ and $W(x) \geq W > 0, W'(0) = W'(1) = 0$. Let $\nu_j$, j=0(1)n, be the approximate solution of (5.5), obtained using (5.28), (5.29). Then, there is a constant $M$ independent of $\varepsilon$ and $h$ such that*

$$\max_j |V(x_j) - \nu_j| \;\leq\; Mh^2$$

## 5.5 Test Example and Numerical Results

In this Section we present some numerical results in support of the theoreticle results:

**Example 5.1** *[66]:*

$$a(x) = 1 + x^2 \ , \ b(x) = \left[\frac{cos(x)}{(3-x)^3}\right] \ , \ f(x) = 4(3x^2 - 3x + 1)\left[\left(x - \frac{1}{2}\right)^2 + 2\right]$$

$$y(0) = -1, \ \ y(1) = 0$$

*Exact Solution : Not available.*

**Example 5.2** *[66]:*

$$a(x) = 1 \ , \ b(x) = 1 \ , \ f(x) = -\left(cos^2\pi x + 2\varepsilon\pi^2 cos2\pi x\right)$$

$$y(0) = 0 \ , \ y(1) = 0$$

*Its exact solution is given by*

$$y(x) = \frac{exp[-(1-x)/\sqrt{\varepsilon}\,] + exp[-x/\sqrt{\varepsilon}\,]}{1 + exp[-1/\sqrt{\varepsilon}\,]} - cos^2\pi x$$

**Example 5.3** *[253]:*

$$a(x) = 1 \ , \ b(x) = 1 + x(1-x)$$

$$f(x) = 1 + x(1-x) + \left[2\sqrt{\varepsilon} - x^2(1-x)\right] exp[-(1-x)/\sqrt{\varepsilon}\,]$$

$$+ \left[2\sqrt{\varepsilon} - x(1-x)^2\right] exp[-x/\sqrt{\varepsilon}\,]$$

$$y(0) = 0 \ , \ y(1) = 0$$

*Its exact solution is given by*

$$y(x) = 1 + (x-1)\ exp[-x/\sqrt{\varepsilon}\,] - x\ exp[-(1-x)/\sqrt{\varepsilon}\,]$$

**Example 5.4** *[186]:*

$$a(x) = 1 \ , \ b(x) = \frac{4}{(x+1)^4}[1 + \sqrt{\varepsilon}\,(x+1)]$$

$$f(x) = -\frac{4}{(x+1)^4}\left[\left\{1 + \sqrt{\varepsilon}\,(x+1) + 4\pi^2\varepsilon\right\} cos\left(\frac{4\pi x}{x+1}\right)\right.$$

$$\left. - 2\pi\varepsilon(x+1) sin\left(\frac{4\pi x}{x+1}\right) + \frac{3\left\{1 + \sqrt{\varepsilon}\,(x+1)\right\}}{1 - exp(-1/\sqrt{\varepsilon}\,)}\right]$$

$$y(0) = 2 \ , \ y(1) = -1$$

*Its exact solution is given by*

$$y(x) = -cos\left(\frac{4\pi x}{x+1}\right) + \frac{3\left\{exp\left(-2x/\sqrt{\varepsilon}\,(x+1)\right) - exp(-1/\sqrt{\varepsilon}\,)\right\}}{1 - exp(-1/\sqrt{\varepsilon}\,)}$$

**Example 5.5** *[210]:*

$$a(x) = 1 \ , \ b(x) = \frac{1}{\varepsilon} \ , \ f(x) = \frac{1}{\varepsilon}\left(x - 1 - x\, exp(-1/\sqrt{\varepsilon}\,)\right)$$

$$y(0) = 0 \ , \ y(1) = 0$$

*Its exact solution is given by*

$$y(x) = x - 1 - x\, exp(-1/\sqrt{\varepsilon}\,) + exp(-x/\sqrt{\varepsilon}\,)$$

**Example 5.6** *[180]:*

$$a(x) = 1 \ , \ b(x) = 1 + x \ , \ f(x) = -40\left[x(x^2 - 1) - 2\varepsilon\right]$$

$$y(0) = 0 \ , \ y(1) = 0$$

*Its exact solution is given by*

$$y(x) = 40x(1 - x)$$

Tables 5.1 and 5.2 contain the results obtained via ST.

Tables 5.5 to 5.16 contain the results obtained via VMCS.

Tables 5.19 to 5.24 contain the results obtained via EFCS.

Results obtained by using equation (5.15) together with equations (5.16), (5.17), (5.18), (5.19), respectively, are denoted by RS1, RS2, RS3 and RS4, whereas ER denotes the results obtained by O'Riordian and Stynes [186] for the corresponding values of $h$ and $\varepsilon$.

Tables 5.1, 5.3 and 5.17 contain the maximum errors (for Example 5.1) based on the double mesh principle as for this example the exact solution is not available.

Tables 5.5 to 5.12, 5.13 to 5.20, 5.23 and 5.24 contain the maximum errors at all the mesh points:

$$\max_j |y(x_j) - \widetilde{\nu}(x_j)|$$

for different $n$ and $\varepsilon$, where $\widetilde{\nu}(x_j)$ is the approximate solution of (5.1) obtained via (5.5) and (5.3).

Table 5.4 contains the numerical rate of uniform convergence:

$$r_{k,\varepsilon} = log_2\left(z_{k,\varepsilon}/z_{k+1,\varepsilon}\right) \quad \text{where} \quad z_{k,\varepsilon} = \max_j |\widetilde{\nu}_j^{h_j/2^k} - \widetilde{\nu}_{2j}^{h_j/2^{k+1}}| \ , \ k = 0, 1, 2, \cdots\cdots\cdots$$

and $\widetilde{\nu}_j^{h_j/2^k}$ denotes the value of $\widetilde{\nu}_j$ for the mesh length $h_j/2^k$. (Tables 5.2, 5.21 and 5.18 have been tabulated with $h_j = h =$ constant as for ST and EFCS we uses of uniform mesh).

Table 5.1: Max. Errors for Example 5.1
Using ST

| $\varepsilon \backslash n$ | | 8 | 16 | 32 | 64 | 128 | 256 | 512 |
|---|---|---|---|---|---|---|---|---|
| 1.0E-0.5 | ER | 0.96E-01 | 0.24E-01 | 0.61E-02 | 0.15E-02 | 0.38E-03 | 0.95E-05 | 0.24E-04 |
| | RS1 | 0.96E-01 | 0.24E-01 | 0.60E-02 | 0.15E-02 | 0.38E-03 | 0.94E-04 | 0.23E-04 |
| | RS2 | 0.48E-01 | 0.12E-01 | 0.30E-02 | 0.76E-03 | 0.19E-03 | 0.47E-04 | 0.12E-04 |
| | RS3 | 0.49E-01 | 0.12E-01 | 0.31E-02 | 0.77E-03 | 0.19E-03 | 0.48E-04 | 0.12E-04 |
| | RS4 | 0.47E-01 | 0.12E-01 | 0.30E-02 | 0.74E-03 | 0.19E-03 | 0.47E-04 | 0.12E-04 |
| 1.0E-1.0 | ER | 0.71E+00 | 0.18E+00 | 0.45E-01 | 0.11E-01 | 0.28E-02 | 0.71E-03 | 0.17E-03 |
| | RS1 | 0.73E+00 | 0.18E+00 | 0.45E-01 | 0.11E-01 | 0.28E-02 | 0.71E-03 | 0.18E-03 |
| | RS2 | 0.37E+00 | 0.93E-01 | 0.23E-01 | 0.58E-02 | 0.15E-02 | 0.37E-03 | 0.91E-04 |
| | RS3 | 0.41E+00 | 0.10E+00 | 0.25E-01 | 0.63E-02 | 0.16E-02 | 0.39E-03 | 0.99E-04 |
| | RS4 | 0.34E+00 | 0.86E-01 | 0.22E-01 | 0.54E-02 | 0.13E-02 | 0.34E-03 | 0.84E-04 |
| 1.0E-1.5 | ER | 0.19E+01 | 0.47E+00 | 0.12E+00 | 0.29E-01 | 0.73E-02 | 0.18E-02 | 0.45E-03 |
| | RS1 | 0.22E+01 | 0.54E+00 | 0.14E+00 | 0.34E-01 | 0.84E-02 | 0.21E-02 | 0.53E-03 |
| | RS2 | 0.12E+01 | 0.29E+00 | 0.73E-01 | 0.18E-01 | 0.46E-02 | 0.11E-02 | 0.29E-03 |
| | RS3 | 0.14E+01 | 0.36E+00 | 0.89E-01 | 0.22E-01 | 0.56E-02 | 0.14E-02 | 0.35E-03 |
| | RS4 | 0.91E+00 | 0.23E+00 | 0.58E-01 | 0.14E-01 | 0.36E-02 | 0.90E-03 | 0.23E-03 |
| 1.0E-0.2 | ER | 0.34E+01 | 0.75E+00 | 0.18E+00 | 0.44E-01 | 0.11E-01 | 0.27E-02 | 0.69E-03 |
| | RS1 | 0.41E+01 | 0.90E+00 | 0.22E+00 | 0.53E-01 | 0.13E-01 | 0.33E-02 | 0.83E-03 |
| | RS2 | 0.19E+01 | 0.48E+00 | 0.12E+00 | 0.30E-01 | 0.75E-02 | 0.19E-02 | 0.47E-03 |
| | RS3 | 0.28E+01 | 0.68E+00 | 0.17E+00 | 0.42E-01 | 0.11E-01 | 0.26E-02 | 0.66E-03 |
| | RS4 | 0.11E+01 | 0.39E+00 | 0.10E+00 | 0.27E-01 | 0.67E-02 | 0.17E-02 | 0.42E-03 |
| 1.0E-0.3 | ER | 0.55E+01 | 0.17E+01 | 0.42E+00 | 0.82E-01 | 0.17E-01 | 0.41E-02 | 0.99E-03 |
| | RS1 | 0.64E+01 | 0.19E+01 | 0.46E+00 | 0.93E-01 | 0.20E-01 | 0.46E-02 | 0.11E-02 |
| | RS2 | 0.13E+01 | 0.50E+00 | 0.17E+00 | 0.43E-01 | 0.11E-01 | 0.26E-02 | 0.65E-03 |
| | RS3 | 0.25E+01 | 0.95E+00 | 0.31E+00 | 0.82E-01 | 0.32E-01 | 0.12E-01 | 0.32E-02 |
| | RS4 | 0.26E+00 | 0.14E+00 | 0.62E-01 | 0.66E-01 | 0.49E-01 | 0.15E-01 | 0.39E-02 |
| 1.0E-0.4 | ER | 0.55E+01 | 0.17E+01 | 0.46E+00 | 0.12E+00 | 0.31E-01 | 0.74E-02 | 0.15E-02 |
| | RS1 | 0.64E+01 | 0.19E+01 | 0.51E+00 | 0.13E+00 | 0.34E-01 | 0.81E-02 | 0.17E-02 |
| | RS2 | 0.97E+00 | 0.33E+00 | 0.97E-01 | 0.28E-01 | 0.87E-02 | 0.27E-02 | 0.73E-03 |
| | RS3 | 0.21E+01 | 0.73E+00 | 0.22E+00 | 0.62E-01 | 0.18E-01 | 0.53E-02 | 0.35E-02 |
| | RS4 | 0.26E-01 | 0.15E-01 | 0.79E-02 | 0.41E-02 | 0.21E-02 | 0.13E-02 | 0.50E-02 |

Table 5.2: Rate of Convergence for Example 5.1
Using ST: $n = 8, 16, 32, 64, 128$

| $\varepsilon \backslash$Rate | | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|---|
| 1.0E-0.5 | ER | 0.198E+01 | 0.200E+01 | 0.200E+01 | 0.200E+01 | 0.200E+01 | 0.200E+01 |
| | RS1 | 0.200E+01 | 0.200E+01 | 0.200E+01 | 0.200E+01 | 0.200E+01 | 0.200E+01 |
| 1.0E-1.0 | ER | 0.199E+01 | 0.200E+01 | 0.200E+01 | 0.200E+01 | 0.200E+01 | 0.200E+01 |
| | RS1 | 0.201E+01 | 0.200E+01 | 0.200E+01 | 0.200E+01 | 0.200E+01 | 0.200E+01 |
| 1.0E-1.5 | ER | 0.202E+01 | 0.201E+01 | 0.200E+01 | 0.200E+01 | 0.200E+01 | 0.201E+01 |
| | RS1 | 0.203E+01 | 0.200E+01 | 0.200E+01 | 0.200E+01 | 0.200E+01 | 0.201E+01 |
| 1.0E-2.0 | ER | 0.220E+01 | 0.207E+01 | 0.202E+01 | 0.200E+01 | 0.200E+01 | 0.206E+01 |
| | RS1 | 0.218E+01 | 0.206E+01 | 0.201E+01 | 0.200E+01 | 0.200E+01 | 0.205E+01 |
| 1.0E-3.0 | ER | 0.174E+01 | 0.199E+01 | 0.234E+01 | 0.224E+01 | 0.210E+01 | 0.208E+01 |
| | RS1 | 0.178E+01 | 0.201E+01 | 0.232E+01 | 0.224E+01 | 0.209E+01 | 0.209E+01 |
| 1.0E-4.0 | ER | 0.173E+01 | 0.186E+01 | 0.193E+01 | 0.197E+01 | 0.206E+01 | 0.191E+01 |
| | RS1 | 0.177E+01 | 0.188E+01 | 0.194E+01 | 0.197E+01 | 0.206E+01 | 0.192E+01 |

Table 5.3: Max. Errors for Example 5.1
Using VMCS: With about 25% mesh points in the boundary layer region
$\delta = 0.001$

| $\varepsilon$ | $n=8$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ |
|---|---|---|---|---|---|---|
| 1/2 | 0.18E-01 | 0.46E-02 | 0.12E-02 | 0.29E-03 | 0.72E-04 | 0.18E-04 |
| 1/4 | 0.36E-01 | 0.92E-02 | 0.23E-02 | 0.57E-03 | 0.14E-03 | 0.36E-04 |
| 1/8 | 0.71E-01 | 0.18E-01 | 0.45E-02 | 0.11E-02 | 0.28E-03 | 0.71E-04 |
| 1/16 | 0.14E+00 | 0.36E-01 | 0.89E-02 | 0.22E-02 | 0.55E-03 | 0.14E-03 |
| 1/32 | 0.27E+00 | 0.69E-01 | 0.17E-01 | 0.43E-02 | 0.11E-02 | 0.27E-03 |
| 1/64 | 0.51E+00 | 0.13E+00 | 0.32E-01 | 0.80E-02 | 0.20E-02 | 0.50E-03 |
| 1/128 | 0.91E+00 | 0.22E+00 | 0.56E-01 | 0.14E-01 | 0.35E-02 | 0.88E-03 |
| 1/256 | 0.15E+01 | 0.36E+00 | 0.90E-01 | 0.22E-01 | 0.56E-02 | 0.14E-02 |

Table 5.4: Rate of convergence for Example 5.1
Using VMCS: With about 25% mesh points in the boundary layer region
$\delta = 0.001, n = 8, 16, 32, 64, 128$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/2 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/4 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/8 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/16 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/32 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/64 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/128 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/256 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |

Table 5.5: Max. Errors for Example 5.2
Using VMCS: With Uniform Mesh

| $\varepsilon$ | $n = 16$ | $n = 32$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ |
|---|---|---|---|---|---|---|
| 1/8 | 0.85E-02 | 0.21E-02 | 0.53E-03 | 0.13E-03 | 0.33E-04 | 0.83E-05 |
| 1/16 | 0.71E-02 | 0.18E-02 | 0.45E-03 | 0.11E-03 | 0.28E-04 | 0.70E-05 |
| 1/32 | 0.57E-02 | 0.14E-02 | 0.36E-03 | 0.89E-04 | 0.22E-04 | 0.56E-05 |
| 1/64 | 0.41E-02 | 0.10E-02 | 0.25E-03 | 0.64E-04 | 0.16E-04 | 0.40E-05 |
| 1/128 | 0.70E-02 | 0.18E-02 | 0.43E-03 | 0.11E-03 | 0.27E-04 | 0.68E-05 |
| 1/256 | 0.16E-01 | 0.38E-02 | 0.93E-03 | 0.23E-03 | 0.58E-04 | 0.15E-04 |
| 1/512 | 0.34E-01 | 0.76E-02 | 0.19E-02 | 0.48E-03 | 0.12E-03 | 0.30E-04 |
| 1/1024 | 0.63E-01 | 0.17E-01 | 0.39E-02 | 0.96E-03 | 0.24E-03 | 0.60E-04 |
| 1/2048 | 0.10E+00 | 0.34E-01 | 0.76E-02 | 0.19E-02 | 0.48E-03 | 0.12E-03 |

Table 5.6: Max. Errors for Example 5.2
Using VMCS: With about 25% mesh points in the boundary layer region
$\delta = O(\sqrt{\varepsilon}\, ln(1/\varepsilon))$

| $\varepsilon$ | $n = 16$ | $n = 32$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ |
|---|---|---|---|---|---|---|
| 1/8 | 0.93E-02 | 0.26E-02 | 0.66E-03 | 0.17E-03 | 0.42E-04 | 0.11E-04 |
| 1/16 | 0.44E-02 | 0.15E-02 | 0.37E-03 | 0.95E-04 | 0.24E-04 | 0.60E-05 |
| 1/32 | 0.48E-02 | 0.13E-02 | 0.33E-03 | 0.84E-04 | 0.21E-04 | 0.53E-05 |
| 1/64 | 0.13E-01 | 0.29E-02 | 0.70E-03 | 0.17E-03 | 0.44E-04 | 0.11E-04 |
| 1/128 | 0.23E-01 | 0.50E-02 | 0.13E-02 | 0.31E-03 | 0.78E-04 | 0.19E-04 |
| 1/256 | 0.32E-01 | 0.71E-02 | 0.18E-02 | 0.44E-03 | 0.11E-03 | 0.28E-04 |
| 1/512 | 0.41E-01 | 0.95E-02 | 0.23E-02 | 0.58E-03 | 0.14E-03 | 0.36E-04 |
| 1/1024 | 0.50E-01 | 0.12E-01 | 0.29E-02 | 0.71E-03 | 0.18E-03 | 0.44E-04 |
| 1/2048 | 0.59E-01 | 0.15E-01 | 0.35E-02 | 0.86E-03 | 0.21E-03 | 0.51E-04 |

Table 5.7: Max. Errors for Example 5.2
Using VMCS: With about 50% mesh points in the boundary layer region
$\delta = O(\sqrt{\varepsilon}\, ln(1/\varepsilon))$

| $\varepsilon$ | $n = 16$ | $n = 32$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ |
|---|---|---|---|---|---|---|
| 1/8 | 0.31E-02 | 0.83E-03 | 0.21E-03 | 0.53E-04 | 0.13E-04 | 0.33E-05 |
| 1/16 | 0.32E-02 | 0.84E-03 | 0.21E-03 | 0.53E-04 | 0.13E-04 | 0.33E-05 |
| 1/32 | 0.38E-02 | 0.95E-03 | 0.24E-03 | 0.60E-04 | 0.15E-04 | 0.37E-05 |
| 1/64 | 0.38E-02 | 0.96E-03 | 0.24E-03 | 0.60E-04 | 0.15E-04 | 0.37E-05 |
| 1/128 | 0.53E-02 | 0.13E-02 | 0.33E-03 | 0.82E-04 | 0.20E-04 | 0.51E-05 |
| 1/256 | 0.73E-02 | 0.19E-02 | 0.46E-03 | 0.11E-03 | 0.29E-04 | 0.71E-05 |
| 1/512 | 0.97E-02 | 0.24E-02 | 0.60E-03 | 0.15E-03 | 0.37E-04 | 0.91E-05 |
| 1/1024 | 0.12E-01 | 0.30E-02 | 0.74E-03 | 0.18E-03 | 0.45E-04 | 0.11E-04 |
| 1/2048 | 0.15E-01 | 0.36E-02 | 0.89E-03 | 0.22E-03 | 0.55E-04 | 0.14E-04 |

Table 5.8: Max. Errors for Example 5.3
Using VMCS: With Uniform Mesh

| $\varepsilon$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ | $n=512$ |
|---|---|---|---|---|---|---|
| 1/8 | 0.15E-02 | 0.36E-03 | 0.91E-04 | 0.23E-04 | 0.57E-05 | 0.14E-05 |
| 1/16 | 0.20E-02 | 0.49E-03 | 0.12E-03 | 0.31E-04 | 0.77E-05 | 0.19E-05 |
| 1/32 | 0.29E-02 | 0.73E-03 | 0.18E-03 | 0.45E-04 | 0.11E-04 | 0.28E-05 |
| 1/64 | 0.51E-02 | 0.13E-02 | 0.31E-03 | 0.78E-04 | 0.20E-04 | 0.49E-05 |
| 1/128 | 0.95E-02 | 0.23E-02 | 0.58E-03 | 0.15E-03 | 0.36E-04 | 0.91E-05 |
| 1/256 | 0.19E-01 | 0.45E-02 | 0.11E-02 | 0.28E-03 | 0.69E-04 | 0.17E-04 |
| 1/512 | 0.38E-01 | 0.85E-02 | 0.21E-02 | 0.53E-03 | 0.13E-03 | 0.33E-04 |
| 1/1024 | 0.67E-01 | 0.18E-01 | 0.42E-02 | 0.10E-02 | 0.26E-03 | 0.64E-04 |
| 1/2048 | 0.11E+00 | 0.36E-01 | 0.81E-02 | 0.20E-02 | 0.50E-03 | 0.13E-03 |

Table 5.9: Max. Errors for Example 5.3
Using VMCS: With about 25% mesh points in the boundary layer region
$\delta = O(\sqrt{\varepsilon}\, ln(1/\varepsilon))$

| $\varepsilon$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ | $n=512$ |
|---|---|---|---|---|---|---|
| 1/8 | 0.60E-02 | 0.15E-02 | 0.38E-03 | 0.94E-04 | 0.24E-04 | 0.59E-05 |
| 1/16 | 0.94E-02 | 0.23E-02 | 0.57E-03 | 0.14E-03 | 0.36E-04 | 0.89E-05 |
| 1/32 | 0.14E-01 | 0.32E-02 | 0.79E-03 | 0.20E-03 | 0.49E-04 | 0.12E-04 |
| 1/64 | 0.19E-01 | 0.43E-02 | 0.11E-02 | 0.26E-03 | 0.65E-04 | 0.16E-04 |
| 1/128 | 0.24E-01 | 0.55E-02 | 0.14E-02 | 0.34E-03 | 0.85E-04 | 0.21E-04 |
| 1/256 | 0.30E-01 | 0.67E-02 | 0.17E-02 | 0.43E-03 | 0.11E-03 | 0.26E-04 |
| 1/512 | 0.37E-01 | 0.84E-02 | 0.21E-02 | 0.52E-03 | 0.13E-03 | 0.32E-04 |
| 1/1024 | 0.44E-01 | 0.10E-01 | 0.25E-02 | 0.63E-03 | 0.16E-03 | 0.38E-04 |
| 1/2048 | 0.51E-01 | 0.13E-01 | 0.30E-02 | 0.74E-03 | 0.18E-03 | 0.44E-04 |

Table 5.10: Max. Errors for Example 5.3
Using VMCS: With about 50% mesh points in the boundary layer region
$\delta = O(\sqrt{\varepsilon}\, ln(1/\varepsilon))$

| $\varepsilon$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ | $n=512$ |
|---|---|---|---|---|---|---|
| 1/8 | 0.17E-02 | 0.42E-03 | 0.11E-03 | 0.26E-04 | 0.66E-05 | 0.16E-05 |
| 1/16 | 0.25E-02 | 0.63E-03 | 0.16E-03 | 0.39E-04 | 0.98E-05 | 0.24E-05 |
| 1/32 | 0.34E-02 | 0.85E-03 | 0.21E-03 | 0.53E-04 | 0.13E-04 | 0.33E-05 |
| 1/64 | 0.46E-02 | 0.11E-02 | 0.28E-03 | 0.70E-04 | 0.17E-04 | 0.44E-05 |
| 1/128 | 0.59E-02 | 0.14E-02 | 0.36E-03 | 0.90E-04 | 0.23E-04 | 0.56E-05 |
| 1/256 | 0.73E-02 | 0.18E-02 | 0.46E-03 | 0.11E-03 | 0.28E-04 | 0.71E-05 |
| 1/512 | 0.88E-02 | 0.23E-02 | 0.56E-03 | 0.14E-03 | 0.35E-04 | 0.86E-05 |
| 1/1024 | 0.11E-01 | 0.27E-02 | 0.74E-03 | 0.20E-03 | 0.50E-04 | 0.13E-04 |
| 1/2048 | 0.13E-01 | 0.36E-02 | 0.10E-02 | 0.30E-03 | 0.77E-04 | 0.19E-04 |

Table 5.11: Max. Errors of O'Riordan and Stynes for Example 5.4

| $n$ | $\varepsilon = (1/n)^{0.25}$ | $\varepsilon = (1/n)^{0.5}$ | $\varepsilon = (1/n)^{0.75}$ | $\varepsilon = (1/n)$ | $\varepsilon = (1/n)^{1.5}$ |
|---|---|---|---|---|---|
| 8 | 0.38E+00 | 0.33E+00 | 0.28E+00 | 0.25E+00 | 0.26E+00 |
| 16 | 0.95E-01 | 0.78E-01 | 0.66E-01 | 0.64E-01 | 0.77E-01 |
| 32 | 0.23E-01 | 0.18E-01 | 0.16E-01 | 0.17E-01 | 0.23E-01 |
| 64 | 0.56E-02 | 0.42E-02 | 0.40E-02 | 0.42E-02 | 0.76E-02 |
| 128 | 0.13E-02 | 0.10E-02 | 0.10E-02 | 0.13E-02 | 0.22E-02 |
| 256 | 0.31E-03 | 0.25E-03 | 0.26E-03 | 0.37E-03 | 0.57E-03 |

Table 5.12: Max. Errors (Ours) for Example 5.4
Using VMCS: With about 25% mesh points in the boundary layer region
$\delta = O(\sqrt{\varepsilon}\, ln(1/\varepsilon))$

| $n$ | $\varepsilon = (1/n)^{0.25}$ | $\varepsilon = (1/n)^{0.5}$ | $\varepsilon = (1/n)^{0.75}$ | $\varepsilon = (1/n)$ | $\varepsilon = (1/n)^{1.5}$ |
|---|---|---|---|---|---|
| 8 | 0.27E+00 | 0.19E+00 | 0.17E+00 | 0.18E+00 | 0.21E+00 |
| 16 | 0.58E-01 | 0.48E-01 | 0.48E-01 | 0.48E-01 | 0.56E-01 |
| 32 | 0.13E-01 | 0.12E-01 | 0.12E-01 | 0.12E-01 | 0.18E-01 |
| 64 | 0.32E-02 | 0.31E-02 | 0.30E-02 | 0.34E-02 | 0.62E-02 |
| 128 | 0.78E-03 | 0.76E-03 | 0.78E-03 | 0.10E-02 | 0.27E-02 |
| 256 | 0.19E-03 | 0.19E-03 | 0.21E-03 | 0.31E-03 | 0.12E-02 |

Table 5.13: Max. Errors for Example 5.5

| | Schatz's : Piecewise Linear [210] | | Schatz's : Hermite Cubics [210] | |
|---|---|---|---|---|
| $\varepsilon$ | $n = 20$ | $n = 40$ | $n = 20$ | $n = 40$ |
| $5^{-1}$ | 0.96E-03 | 0.24E-03 | 0.34E-05 | 0.26E-06 |
| $5^{-2}$ | 0.27E-01 | 0.60E-02 | 0.83E-03 | 0.90E-04 |
| $5^{-3}$ | 0.21E+00 | 0.12E+00 | 0.33E-01 | 0.94E-02 |
| $5^{-4}$ | 0.26E+00 | 0.26E+00 | 0.78E-01 | 0.68E-01 |
| $5^{-5}$ | 0.27E+00 | 0.27E+00 | 0.82E-01 | 0.82E-01 |
| $5^{-6}$ | 0.27E+00 | 0.27E+00 | 0.82E-01 | 0.82E-01 |

Table 5.14: Max. Errors for Example 5.5
Using VMCS: $\delta = O(\varepsilon\, ln(1/\varepsilon))$

| | Ours : Unif. Mesh | | About 25% mesh points in BL | | About 50% mesh points in BL | |
|---|---|---|---|---|---|---|
| $\varepsilon$ | $n = 20$ | $n = 40$ | $n = 20$ | $n = 40$ | $n = 20$ | $n = 40$ |
| $5^{-1}$ | 0.96E-03 | 0.24E-03 | 0.44E-02 | 0.71E-03 | 0.93E-03 | 0.23E-03 |
| $5^{-2}$ | 0.27E-01 | 0.60E-02 | 0.25E-01 | 0.37E-02 | 0.80E-02 | 0.23E-02 |
| $5^{-3}$ | 0.21E+00 | 0.12E+00 | 0.54E-01 | 0.83E-02 | 0.16E-01 | 0.90E-02 |
| $5^{-4}$ | 0.27E+00 | 0.26E+00 | 0.85E-01 | 0.16E-01 | 0.16E-01 | 0.65E-02 |
| $5^{-5}$ | 0.27E+00 | 0.27E+00 | 0.11E+00 | 0.25E-01 | 0.25E-01 | 0.56E-02 |
| $5^{-6}$ | 0.27E+00 | 0.27E+00 | 0.14E+00 | 0.34E-01 | 0.34E-01 | 0.71E-02 |

Table 5.15: Max. Errors for Example 5.6
Using VMCS: $n = 16$

| $\varepsilon$ | Miller [165] | Niijima [179] | Niijima [180] | Our Results (Unif. Mesh) |
|---|---|---|---|---|
| 0.1E-03 | 0.25E-1 | 0.26E-1 | 0.65E-4 | 0.36E-14 |
| 0.1E-04 | 0.21E-1 | 0.24E-1 | 0.36E-4 | 0.36E-14 |
| 0.1E-05 | 0.70E-2 | 0.17E-1 | 0.33E-4 | 0.36E-14 |
| 0.1E-06 | 0.75E-3 | 0.69E-2 | 0.26E-4 | 0.27E-14 |
| 0.1E-07 | 0.74E-4 | 0.23E-2 | 0.20E-4 | 0.36E-14 |
| 0.1E-08 | 0.67E-5 | 0.76E-3 | 0.20E-4 | 0.36E-14 |
| 0.1E-09 | 0.00E+0 | 0.24E-3 | 0.11E-4 | 0.36E-14 |

Table 5.16: Max. Errors for Example 5.6
Using VMCS: $n = 32$

| $\varepsilon$ | Miller [165] | Niijima [179] | Niijima [180] | Our Results (Unif. Mesh) |
|---|---|---|---|---|
| 0.1E-03 | 0.64E-2 | 0.65E-2 | 0.59E-4 | 0.71E-14 |
| 0.1E-04 | 0.61E-2 | 0.64E-2 | 0.21E-4 | 0.36E-14 |
| 0.1E-05 | 0.41E-2 | 0.56E-2 | 0.35E-4 | 0.36E-14 |
| 0.1E-06 | 0.77E-3 | 0.31E-2 | 0.39E-4 | 0.36E-14 |
| 0.1E-07 | 0.76E-4 | 0.12E-2 | 0.21E-4 | 0.36E-14 |
| 0.1E-08 | 0.67E-5 | 0.38E-3 | 0.21E-4 | 0.36E-14 |
| 0.1E-09 | 0.00E+0 | 0.13E-3 | 0.14E-4 | 0.27E-14 |

Table 5.17: Max. Errors for Example 5.1
Using EFCS

| $\varepsilon$ | $n = 8$ | $n = 16$ | $n = 32$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ |
|---|---|---|---|---|---|---|---|
| 1/4 | 0.20E-01 | 0.49E-02 | 0.12E-02 | 0.31E-03 | 0.77E-04 | 0.19E-04 | 0.48E-05 |
| 1/8 | 0.39E-01 | 0.96E-02 | 0.24E-02 | 0.60E-03 | 0.15E-03 | 0.38E-04 | 0.94E-05 |
| 1/16 | 0.75E-01 | 0.19E-01 | 0.47E-02 | 0.12E-02 | 0.29E-03 | 0.73E-04 | 0.18E-04 |
| 1/32 | 0.14E+00 | 0.35E-01 | 0.88E-02 | 0.22E-02 | 0.55E-03 | 0.14E-03 | 0.34E-04 |
| 1/64 | 0.25E+00 | 0.63E-01 | 0.16E-01 | 0.40E-02 | 0.99E-03 | 0.25E-03 | 0.62E-04 |
| 1/128 | 0.42E+00 | 0.11E+00 | 0.26E-01 | 0.66E-02 | 0.16E-02 | 0.41E-03 | 0.10E-03 |
| 1/256 | 0.64E+00 | 0.16E+00 | 0.40E-01 | 0.99E-02 | 0.25E-02 | 0.62E-03 | 0.15E-03 |

Table 5.18: Rate of Convergence for Example 5.1
Using EFCS
$n = 8, 16, 32, 64, 128$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/4 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/8 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/16 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/32 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/64 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/128 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/256 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |

Table 5.19: Max. Errors for Example 5.3
Using EFCS, Without Using Fitting Factor

| $\varepsilon$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ | $n=512$ |
|---|---|---|---|---|---|---|
| 1/32 | 0.29E-02 | 0.73E-03 | 0.18E-03 | 0.45E-04 | 0.11E-04 | 0.28E-05 |
| 1/64 | 0.51E-02 | 0.13E-02 | 0.31E-03 | 0.78E-04 | 0.20E-04 | 0.49E-05 |
| 1/128 | 0.95E-02 | 0.23E-02 | 0.58E-03 | 0.15E-03 | 0.36E-04 | 0.91E-05 |
| 1/256 | 0.19E-01 | 0.45E-02 | 0.11E-02 | 0.28E-03 | 0.69E-04 | 0.17E-04 |
| 1/512 | 0.38E-01 | 0.85E-02 | 0.21E-02 | 0.53E-03 | 0.13E-03 | 0.33E-04 |
| 1/1024 | 0.67E-01 | 0.18E-01 | 0.42E-02 | 0.10E-02 | 0.26E-03 | 0.64E-04 |
| 1/2048 | 0.11E+00 | 0.36E-01 | 0.81E-02 | 0.20E-02 | 0.50E-03 | 0.13E-03 |

Table 5.20: Max. Errors for Example 5.3
Using EFCS: Using Fitting Factor

| $\varepsilon$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ | $n=512$ |
|---|---|---|---|---|---|---|
| 1/32 | 0.40E-03 | 0.99E-04 | 0.25E-04 | 0.62E-05 | 0.15E-05 | 0.39E-06 |
| 1/64 | 0.53E-03 | 0.13E-03 | 0.33E-04 | 0.82E-05 | 0.21E-05 | 0.51E-06 |
| 1/128 | 0.83E-03 | 0.19E-03 | 0.46E-04 | 0.12E-04 | 0.29E-05 | 0.72E-06 |
| 1/256 | 0.13E-02 | 0.26E-03 | 0.66E-04 | 0.16E-04 | 0.41E-05 | 0.10E-05 |
| 1/512 | 0.18E-02 | 0.42E-03 | 0.95E-04 | 0.23E-04 | 0.58E-05 | 0.14E-05 |
| 1/1024 | 0.25E-02 | 0.62E-03 | 0.13E-03 | 0.33E-04 | 0.81E-05 | 0.20E-05 |
| 1/2048 | 0.33E-02 | 0.88E-03 | 0.21E-03 | 0.47E-04 | 0.12E-04 | 0.29E-05 |

Table 5.21: Rate of Convergence for Example 5.3
Using EFCS: Using Fitting Factor
$n = 16, 32, 64, 128, 256$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1/32 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/64 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/128 | 0.22E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/256 | 0.23E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.21E+01 |
| 1/512 | 0.22E+01 | 0.22E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.21E+01 |
| 1/1024 | 0.21E+01 | 0.23E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.21E+01 |
| 1/2048 | 0.21E+01 | 0.22E+01 | 0.22E+01 | 0.20E+01 | 0.20E+01 | 0.21E+01 |

Table 5.22: Max. Errors for Example 5.4
Obtained by O'Riordan and Stynes

| $n$ | $\varepsilon = 1$ | $\varepsilon = (1/n)^{0.25}$ | $\varepsilon = (1/n)^{0.5}$ | $\varepsilon = (1/n)^{0.75}$ | $\varepsilon = (1/n)$ |
|---|---|---|---|---|---|
| 8 | 0.42E+00 | 0.38E+00 | 0.33E+00 | 0.28E+00 | 0.25E+00 |
| 16 | 0.11E+00 | 0.95E-01 | 0.78E-01 | 0.66E-01 | 0.64E-01 |
| 32 | 0.27E-01 | 0.23E-01 | 0.18E-01 | 0.16E-01 | 0.17E-01 |
| 64 | 0.69E-02 | 0.56E-02 | 0.42E-02 | 0.40E-02 | 0.42E-02 |
| 128 | 0.17E-02 | 0.13E-02 | 0.10E-02 | 0.10E-02 | 0.13E-02 |
| 256 | 0.43E-03 | 0.31E-03 | 0.25E-03 | 0.26E-03 | 0.37E-03 |
| 512 | 0.11E-03 | 0.74E-04 | 0.63E-04 | 0.73E-04 | 0.10E-03 |

Table 5.23: Max. Errors for Example 5.4
Ours: Using EFCS

| $n$ | $\varepsilon = 1$ | $\varepsilon = (1/n)^{0.25}$ | $\varepsilon = (1/n)^{0.5}$ | $\varepsilon = (1/n)^{0.75}$ | $\varepsilon = (1/n)$ |
|---|---|---|---|---|---|
| 8 | 0.20E+00 | 0.18E+00 | 0.14E+00 | 0.10E+00 | 0.10E+00 |
| 16 | 0.54E-01 | 0.47E-01 | 0.38E-01 | 0.26E-01 | 0.22E-01 |
| 32 | 0.14E-01 | 0.12E-01 | 0.95E-02 | 0.64E-02 | 0.92E-02 |
| 64 | 0.35E-02 | 0.30E-02 | 0.24E-02 | 0.16E-02 | 0.27E-02 |
| 128 | 0.86E-03 | 0.76E-03 | 0.60E-03 | 0.40E-03 | 0.66E-03 |
| 256 | 0.22E-03 | 0.19E-03 | 0.15E-03 | 0.10E-03 | 0.16E-03 |
| 512 | 0.54E-04 | 0.47E-04 | 0.37E-04 | 0.25E-04 | 0.41E-04 |

Table 5.24: Max. Errors for Example 5.5
Using EFCS

| | Schatz's : Piecewise Linear [210] | | Schatz's : Hermite Cubics [210] | | Our Results | |
|---|---|---|---|---|---|---|
| $\varepsilon$ | $n = 20$ | $n = 40$ | $n = 20$ | $n = 40$ | $n = 20$ | $n = 40$ |
| $5^{-1}$ | 0.96E-03 | 0.24E-03 | 0.34E-05 | 0.26E-06 | 0.18E-14 | 0.60E-14 |
| $5^{-2}$ | 0.27E-01 | 0.60E-02 | 0.83E-03 | 0.90E-04 | 0.33E-15 | 0.56E-15 |
| $5^{-3}$ | 0.21E+00 | 0.12E+00 | 0.33E-01 | 0.94E-02 | 0.22E-15 | 0.22E-15 |
| $5^{-4}$ | 0.26E+00 | 0.26E+00 | 0.78E-01 | 0.68E-01 | 0.22E-15 | 0.22E-15 |
| $5^{-5}$ | 0.27E+00 | 0.27E+00 | 0.82E-01 | 0.82E-01 | 0.11E-15 | 0.11E-15 |
| $5^{-6}$ | 0.27E+00 | 0.27E+00 | 0.82E-01 | 0.82E-01 | 0.11E-15 | 0.11E-15 |

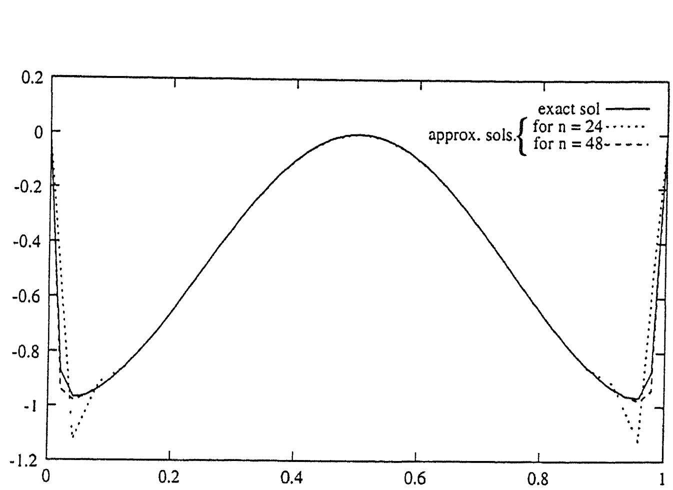


Figure 5.1: Exact and Approx. Solutions of Example 5.2
for $\varepsilon = 0.0001$ and $n = 24, 48$ with Uniform Mesh

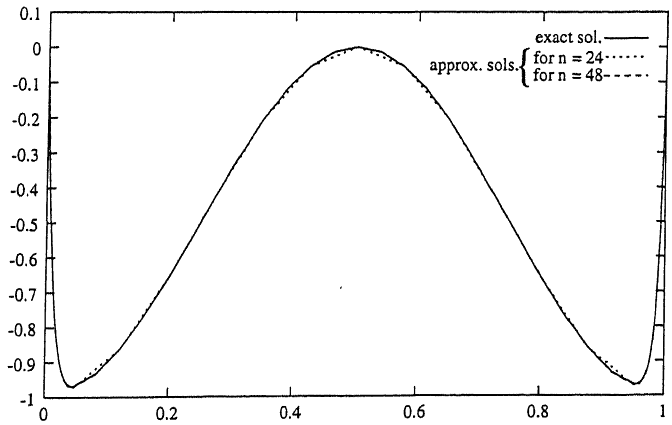


Figure 5.2: Exact and Approx. Solutions of Example 5.2
for $\varepsilon = 0.0001$ and $n = 24, 48$ with about 50 % points in the B. Layer

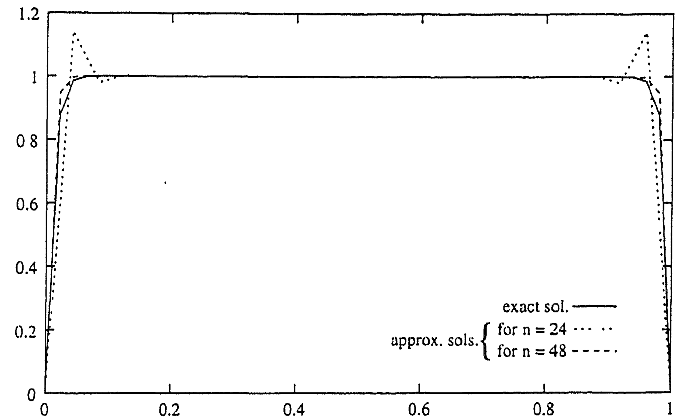


Figure 5.3: Exact and Approx. Solutions of Example 5.3 for $\varepsilon = 0.0001$ and $n = 24, 48$ with Uniform Mesh

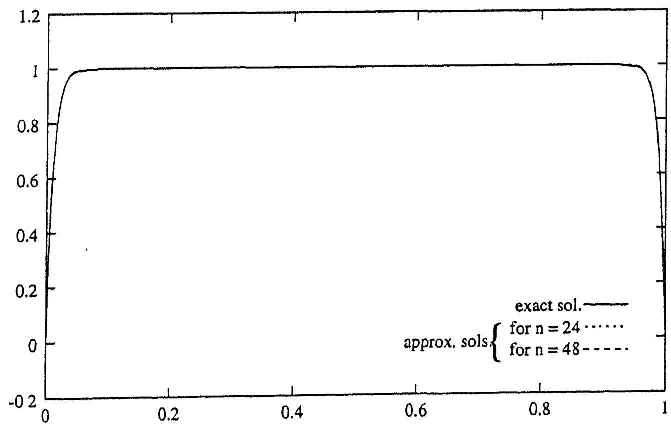


Figure 5.4: Exact and Approx. Solutions of Example 5.3 for $\varepsilon = 0.0001$ and $n = 24, 48$ with about 50 % points in the B. Layer

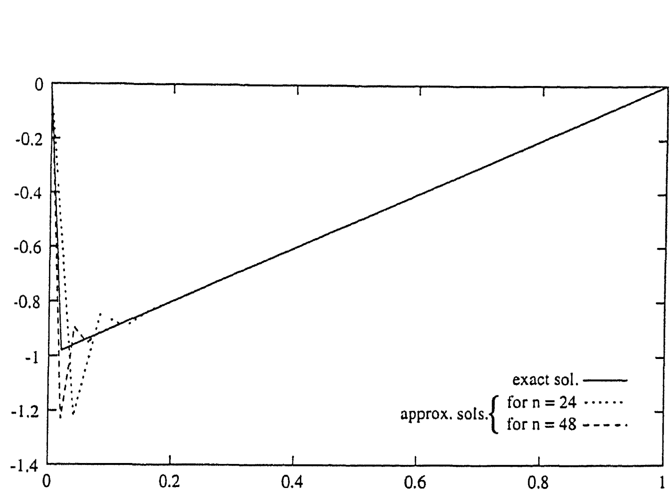


Figure 5.5: Exact and Approx. Solutions of Example 5.5
for $\varepsilon = 1/625$ and $n = 24, 48$ with Uniform Mesh

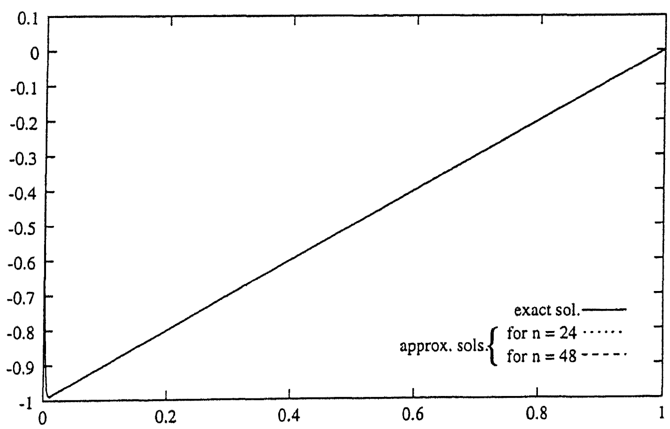


Figure 5.6: Exact and Approx. Solutions of Example 5.5
for $\varepsilon = 1/625$ and $n = 24, 48$ with about 50 % points in the B. Layer

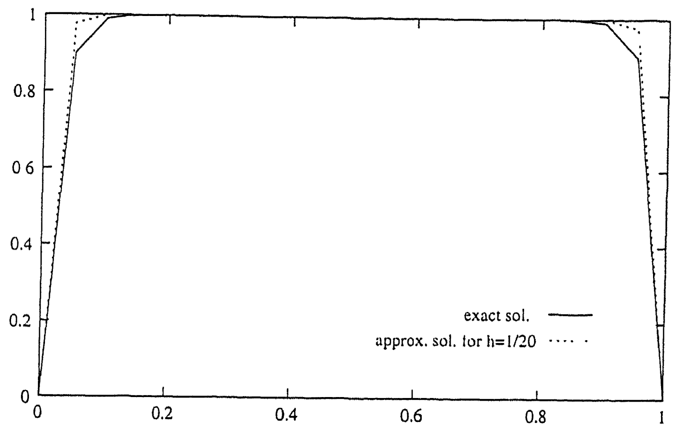


Figure 5.7: Exact and Approx. Solutions of Example 5.3 for $\varepsilon = 0.0005$ and $n = 20$ Without Using Fitting Factor

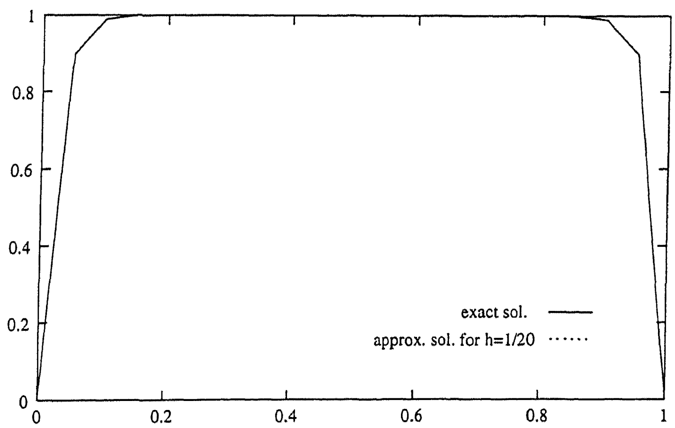


Figure 5.8: Exact and Approx. Solutions of Example 5.3 for $\varepsilon = 0.0005$ and $n = 20$ With Using Fitting Factor

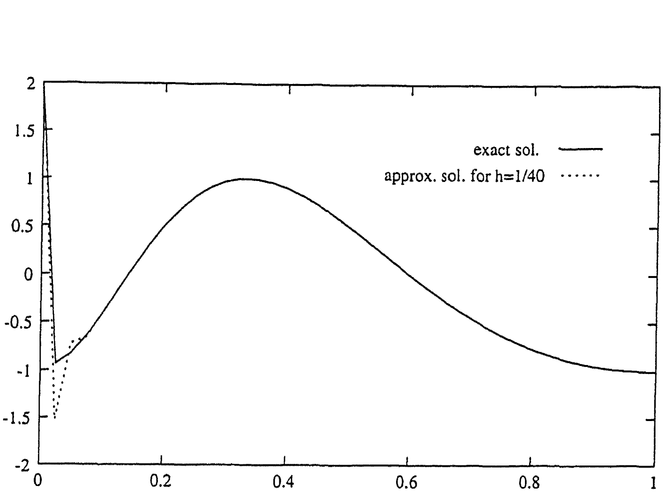


Figure 5.9: Exact and Approx. Solutions of Example 5.4 for $\varepsilon = 0.0001$ and $n = 40$ Without Using Fitting Factor

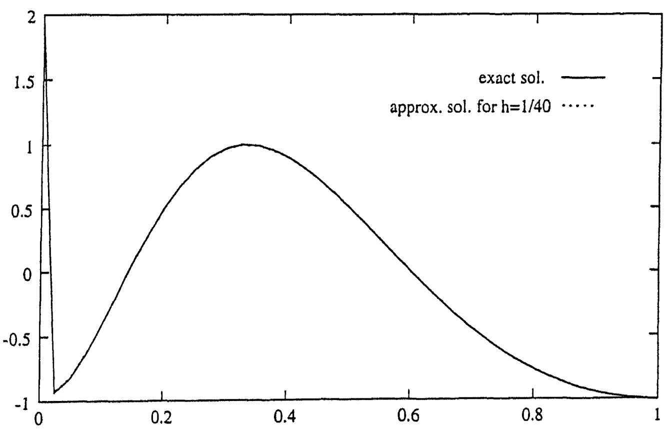


Figure 5.10: Exact and Approx. Solutions of Example 5.4 for $\varepsilon = 0.0001$ and $n = 40$ With Using Fitting Factor

## 5.6 Discussion

We have described three numerical methods for solving self-adjoint singular perturbation problems using spline in tension and cubic spline. All these three methods have been analysed for convergence. Several numerical examples have been solved to demonstrate the applicability of the proposed method.

It can be seen from the Table 5.1 that the results obtained by our method using (5.16) are almost same as those of O'Riordan and Stynes, whereas, the results obtained by using (5.17), (5.18) and (5.19) are better than those of the O'Riordan and Stynes. In some cases when $W(0)$ is not defined then scheme (5.19) will be useful.

In the given mesh selection strategy (in the method VMCS), the boundary layer width $\delta$ plays an important role. According to Miller et al. [168], if the solution of the homogeneous singular perturbation problem involves the functions of the type $exp(-x/\varepsilon)$, then $\delta = O(\varepsilon\, ln\,(1/\varepsilon))$ whereas in case the solution involves the functions of the type $exp(-x/\sqrt{\varepsilon}\ )$, then $\delta = O(\sqrt{\varepsilon}\, ln(1/\varepsilon))$. We used this fact while solving Examples 5.2, 5.3, 5.4 and 5.5. For Examples 5.2, 5.3 and 5.4 we took $\delta = O(\sqrt{\varepsilon}\, ln(1/\varepsilon))$ whereas for Example 5.5, we took $\delta = O(\varepsilon\, ln\,(1/\varepsilon))$ (see [210] as well as using the above argument of Miller et al). For Example 5.1, the exact solution is not available. Hence we took $\delta = 0.001$ and computed the results. Also the exact solution of Example 5.6 is smooth, therefore we took uniform mesh throughout the region.

Our mesh selection procedure needs prior knowledge of $\delta$, $\widetilde{h}_1$ and $K$. We have chosen $\delta$ as above whereas the other two parameters have been taken as $\widetilde{h}_1 = 0.001$ and $K = 1$. However the increase in the value of $K$ will lead to more concentration of points near the boundaries. Moreover, for a fixed $K$, the increase in the value of $\widetilde{h}_1$ leads to the same conclusion.

As is seen from the Tables 5.5 - 5.7 and 5.8 - 5.10, the results on non-equidistant grid are better for smaller $\varepsilon$ for Examples 5.2 and 5.3. Example 5.4 has been solved earlier by O'Riordan and Stynes [186]. We obtained comparable results to those of O'Riordan and Stynes (see Tables 5.11 and 5.12). Using finite element techniques, Example 5.5 has been solved earlier by Schatz and Wahlbin [210]. Our results with uniform mesh and with about 25% and 50% mesh points in the boundary layer regions together with

their results are presented in Tables 5.13 and 5.14, respectively. By taking about 50% mesh points in the boundary layer regions, we obtain better results for smaller $\varepsilon$ than those in [210]. Similarly, for Example 5.6 we have compared our results with those of Miller [165], Niijima [179] and Niijima [180] and obtained better results which can be seen from the Tables 5.15 and 5.16.

To further corroborate the applicability of the proposed methods (VMCS and EFCS), graphs have been plotted for some Examples for values of $x \in [0, 1]$ versus the computed (termed as approximate) solutions obtained at different values of $x$ for a fixed $\varepsilon$. For each plot we took $n = 24$ and 48. Figure 5.1 is the graph, which is plotted using uniform mesh throughout the region, for Example 5.2 for $\varepsilon = 0.0001$, whereas Figure 5.2 is the graphs which is plotted using about 50% mesh points in the boundary layer regions (according to the given mesh selection strategy) for the epsilon value 0.0001 and $\delta = O(\sqrt{\varepsilon}\, ln(1/\varepsilon))$. Similarly, Figure 5.3 is the graph which is plotted using uniform mesh and 5.4 is the graphs which is plotted using about 50% mesh points in the boundary layer regions for the similar values of $\varepsilon$ and $\delta$ as were in the case of figures 5.1 - 5.2. Figure 5.5 is the graph, using uniform mesh throughout the region, for Example 5.5 for $5^{-4}$ whereas Figure 5.6 is the graph which is plotted using about 50% mesh points in the boundary layer region for the epsilon value $5^{-4}$ and $\delta = O(\varepsilon\, ln\,(1/\varepsilon))$. It can be seen from the figures 5.1, 5.3 and 5.5 that the exact and approximate solutions with uniform mesh are identical for most of the range except the boundary layer regions where these two solutions deviate from each other for smaller $\varepsilon$. To control these fluctuations, we took more mesh points in the boundary layer and the resulting behaviour can be seen from the figures 5.2, 5.4 and 5.6, respectively, for the corresponding values of $\varepsilon$. The similar observation can be made for other Examples also.

For the graphs which are plotted using EFCS, we took $n = 20$ and 40 for Example 5.3 and 5.4, respectively. Figure 5.7 is the graph without using fitting factor for Example 5.3 for $\varepsilon = 0.0005$, whereas Figure 5.8 is the graph which is plotted using fitting factor for the same value of $n$ and $\varepsilon = 0.0005$. Similarly Figure 5.9 is the graph which is plotted without using fitting factor for Example 5.4 for $\varepsilon = 0.0001$ whereas Figure 5.10 is the graph which is plotted using fitting factor for the same value of $n$ and $\varepsilon = 0.0001$. It can

be seen from Figures 5.7 and 5.9 that the exact and approximate solutions without using fitting factor are deviate from each other in the boundary layer regions for smaller $\varepsilon$. To control these fluctuations, we used fitting factor technique and the resulting behaviour of these two solutions can be seen from the Figures 5.8 and 5.10. The similar observation can be made for the other examples also.

We have copmputed the rate of convergence using all the three methods for few examples. The results are contained in the Tables 5.2, 5.4, 5.21 and 5.18. The same can be seen for other Examples also.

As is seen from the Tables 5.19 and 5.20, the results obtained using fitting factor are better than those without using fitting factor.

Example 5.4 has been solved earlier by O'Riordan and Stynes [186]. We obtain better results than those in [186]. Using finite element techniques Example 5.5 has been solved by Schatz and Wahlbin [210]. Our results are far better than those in [210].

While using the fitting factor technique, we have replaced $\varepsilon$ by $\sigma(x, \varepsilon)$ in the normalized form and not in the original self-adjoint problem (i.e. problem (5.1)), with $a(x) \neq$ constant because in that case $\varepsilon$ is a multiple of both the second and first derivative terms which will cause implicit expressions whereas in normalized form $\varepsilon$ is multiplying with the second derivative term only and hence the fitting factor technique on the normalized form can easily be implemented. This shows the importance of reducing the original self-adjoint problem to normal form.

Finally we would like to remark that if we do not transform problem (5.1) to the normal form and if we discretize (5.1) directly to the normal form then after getting the tridiagonal system one can observe that out of the two diagonal elements, viz., super- and sub-diagonal elements one is exponentially large whereas the other is exponentially small as the parameter $\varepsilon \to 0$ which causes the illconditioning in the system. This happens when $a(x)$ is not a constant. That is why we need to transform problem (5.1) to the normal form.

We also remark that the two procedures of getting the improved results, viz., the variable mesh method and the exponentially fitted method, may be considered as alternative methods. However, if someone likes to solve the problem on relatively coarser mesh then the exponentially fitted method will be more useful.

# Chapter 6

## SINGULARLY PERTURBED NON-LINEAR TWO-POINT BVPs

## 6.1 Introduction

Consider the singularly perturbed problem

$$Ly \equiv \varepsilon y'' = F(x, y, y') \tag{6.1}$$

$$y(0) = \eta_0 \ , \ y(1) = \eta_1 \tag{6.2}$$

where, $\eta_0, \eta_1 \in R$, $0 < x < 1$, $\varepsilon$ is a small positive parameter and assume that $F$ is a smooth function satisfying:

(i) $\frac{\partial}{\partial z}F(x, y, z) \leq 0$

(ii) $\frac{\partial}{\partial y}F(x, y, z) \geq 0$

(iii) $\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial z}\right) F(x, y, z) \geq \alpha > 0$ and

(iv) the growth condition

$$F(x, y, z) = O\left(|z|^2\right) \quad as \quad z \to \infty.$$

for all $x \in [0, 1]$ and all real $y$ and $z$. Under these conditions, the problem (6.1)-(6.2) has a unique solution [100].

These type of problems arise frequently in applications including geophysical fluid dynamics, in the investigation of oceanic and atmospheric circulation [39], the theory of nonpremixed combustion [49], geodynamics [65], [92], chemical reactions [263], optimal

control [11], etc. General results about the analytical solution of such problems are known and the boundary layer behaviour of the solution $y(x, \varepsilon)$ has been intensively investigated. Many basic results can be found in [49], [101] and [182].

Several methods have been developed for the numerical solution of nonlinear singularly perturbed boundary value problems, with particular attention to problems with one or two boundary layers at the ends of the underlying interval. We mention works based on finite differences ([3], [66], [139], [199], [256], [260], [263]), finite element ([29]), collocation ([14], [160]) and multiple shooting ([157]). Most of these workers use suitable subdivison of the underlying interval, with nodes suffenciently dense in the boundary layer regions. The mesh selection strategy is not trivial and involves theoretical and pratical problems closely depeding on the behaviour of $y(x, \varepsilon)$. Other numerical results can be found in [44], [46], [266], etc.

In this chapter we have presented three approaches (based on Spline in Compression referred as SC, Spline in Tension referred as ST and Variable Mesh Cubic Spline referred as VMCS). SC and ST are the generalizations of the methods developed for linear problems in chapters 2 and 3, respectively. Using the well known quasilinearization method of Bellman and Kalaba [23], the original nonlinear differential equation has been linearized in the sequence of linear differential equations. Each of these linear equations is then solved by the schemes derived using different splines. In limit, the solution of these linearized equations converges to the solution of the original nonlinear equation. By making use of the continuity of the first order derivative of the spline functions, the resulting spline difference schemes gives a tridiagonal system which can be solved efficiently by the well known algorithms.

In Section 6.2 we give a brief description of the methods. Splines of various kinds used in this chapter are defined in Section 6.3. The derivation of the difference scheme has been given in Section 6.4. In Section 6.5 we have shown the second order accuracy of the methods. In Section 6.6 we have solved some numerical examples to demonstrate the applicability of the proposed method. The discussion on our results is given in Section 6.7.

## 6.2 Description of the Methods

We adopt the quasilinearization process for solving nonlinear two-point boundary value problems by the following steps:

(i) Linearizing the nonlinear ordinary differential equations around a nominal solution, which satisfies the specified boundary conditions.

(ii) Solving a sequence of linear two-point boundary value problems in which the solution of the $k^{th}$ linear two-point boundary value problem satisfies the specified boundary conditions and is taken as the nominal profile for the $(k+1)^{st}$ linear two-point boundary value problem.

We spell out in detail the quasilinearization process by considering the singularly perturbed second order nonlinear ordinary differential equation (6.1).

Suppose that we have $y^{(k)}(x)$, $(y')^{(k)}(x)$ as the $k^{th}$ nominal solution to (6.1) over the interval [0,1], which is nominal in the sense that the boundary conditions (6.2) are satisfied exactly but the profiles $y^{(k)}(x)$, $(y')^{(k)}(x)$ only satisfy the differential equation (6.1) approximately.

Expanding the right hand side of (6.1) in a Taylor's series up to first-order terms around the nominal solution $y^{(k)}(x), (y')^{(k)}(x)$; we have

$$\begin{aligned}\varepsilon(y'')^{k+1} &\approx F\left(y^{(k)}(x),(y')^{(k)}(x)\right)+\left(\frac{\partial F}{\partial y}\right)\left(y^{(k+1)}(x)-y^{(k)}(x)\right)\\ &\quad+\left(\frac{\partial F}{\partial y'}\right)\left((y')^{(k+1)}(x)-(y')^{(k)}(x)\right) \qquad (6.3)\end{aligned}$$

$$\begin{aligned}\Rightarrow\quad &\varepsilon(y'')^{k+1}+\left(-\frac{\partial F}{\partial y'}\right)(y')^{(k+1)}(x)+\left(-\frac{\partial F}{\partial y}\right)y^{(k+1)}(x)\\ &=F\left(y^{(k)}(x),(y')^{(k)}(x)\right)-\left(\frac{\partial F}{\partial y'}\right)(y')^{(k)}(x)-\left(\frac{\partial F}{\partial y}\right)y^{(k)}(x) \qquad (6.4)\end{aligned}$$

Equation (6.4) is linear in $y^{(k+1)}(x)$.

Instead of solving the nonlinear two-point boundary value problem (6.1) with the boundary conditions (6.2), we now solve a sequence of linear two-point boundary value problems (6.4) for $k = 0, 1, 2, ...$ (by different kind of Splines, depending on the sign of the coefficients of the first derivative and the function terms in the quasilinearization process

equations. These splines are defined in the next section). The boundary conditions are (6.5):

$$y^{(k)}(0) = y(0) = \eta_0 \quad ; \quad y^{(k)}(1) = y(1) = \eta_1 \tag{6.5}$$

Theoretically, for a solution to the nonlinear problem we require

$$\lim_{k\to\infty} y^{(k)}(x) = y^*(x) \ , \ \ 0 \le x \le 1$$

where $y^*(x)$ is the solution of the nonlinear problem.

Numerically we require that

$$\left|y^{(k+1)}(x) - y^{(k)}(x)\right| < \text{Tol.} \ , \ \ 0 \le x \le 1$$

where Tol. is a small tolerance prescribed by us. If the tolerance test is passed, we terminate the calculations and the profile $y^{(k+1)}(x)$ is the numerical solution to the nonlinear two-point boundary value problem.

## 6.3 Different Kind of Splines

Let $a(x)$ and $b(x)$ are the coefficients of $(y')^{(k+1)}(x)$ and $y^{(k+1)}(x)$ in (6.4) and denote the right hand side of (6.4) by $f(x)$. Now we define the three splines as follows:

### 6.3.1 Spline in Compression (SC)

Consider the following singularly perturbed linear two-point bondary value problem:

$$\left.\begin{array}{l} Ly \equiv \varepsilon y'' + a(x)y' + b(x)y = f(x) \\ y(0) = \eta_0 \ , \ y(1) = \eta_1 \ ; \ \eta_0, \eta_1 \in R \end{array}\right\} \tag{6.6}$$

where $a(x) > c > 0$, $b(x) > 0$, $c$ is some constant and $\varepsilon$ is a small positive parameter.

For $x \in [x_{j-1}, x_j]$, we define

$$\alpha_j = \frac{a_{j-1}+a_j}{2} \ , \ \beta_j = \frac{b_{j-1}+b_j}{2} \ \text{and} \ \gamma_j = \frac{f_{j-1}+f_j}{2}$$

The approximate solution of this problem, is sought in the form of the function $S(x)$, which on each interval $[x_{j-1}, x_j]$, (denoted by $S_j(x)$) satisfies the following relations:-

(i) the differential equation

$$\varepsilon S_j''(x) + \alpha_j S_j'(x) + \beta_j S_j(x) = \gamma_j \tag{6.7}$$

(ii) the interpolating conditions

$$S_j(x_{j-1}) = u_{j-1}, \quad S_j(x_j) = u_j \tag{6.8}$$

(iii) the continuity condition

$$S_j'(x_j^+) = S_j'(x_j^-) \tag{6.9}$$

(iv) the consistency condition

$$\frac{p_j}{2} = tan\frac{p_j}{2}, \ \frac{t_j}{2} = tan\frac{t_j}{2} \ ; \ p_j = \frac{h\alpha_j}{2\varepsilon}, \ t_j = \frac{h(\alpha_j^2 - 4\beta_j\varepsilon)^{1/2}}{2\varepsilon} \tag{6.10}$$

where,

$$x \in [x_{j-1}, x_j], \ x_j = jh, \ j = 1(1)n, \ h = 1/n.$$

Solving equation (6.7) with the help of (6.8), we obtain

$$S_j(x) = \frac{1}{-sinh\ g_j h} exp\left(-\frac{\alpha_j x}{2\varepsilon}\right)[D_j\ sinh\ g_j(x_{j-1} - x) + E_j\ sinh\ g_j(x - x_j)] + \frac{\gamma_j}{\beta_j} \tag{6.11}$$

where,

$$D_j = \left(u_j - \frac{\gamma_j}{\beta_j}\right) exp\left(\frac{\alpha_j x_j}{2\varepsilon}\right)$$

$$E_j = \left(u_{j-1} - \frac{\gamma_j}{\beta_j}\right) exp\left(\frac{\alpha_j x_{j-1}}{2\varepsilon}\right) \ , \ g_j = \frac{\sqrt{\alpha_j^2 - 4\beta_j\varepsilon}}{2\varepsilon}$$

Equation (6.11) together with (6.10) is known as spline in compression [110]. We will use this spline function to derive the difference scheme in next section.

**Remark 1**: If $b(x) \equiv 0$, then instead of solving (6.6), we will solve

$$\left.\begin{aligned} Ly \equiv \varepsilon y'' + a(x)y' &= f(x) \\ y(0) = \eta_0 \ , \ y(1) &= \eta_1 \end{aligned}\right\} \tag{6.12}$$

and therefore in this case the corresponding differential equation and the consistency condition will be

$$\varepsilon S_j''(x) + \alpha_j S_j'(x) = \gamma_j \tag{6.13}$$

$$\frac{p_j}{2} = tan\frac{p_j}{2}\ ,\ p_j = \frac{h\alpha_j}{\varepsilon} \tag{6.14}$$

and the solution of (6.13)-(6.8) will be

$$\begin{aligned} S_j(x) &= \frac{1}{H_j}\left[F_j\, exp\left(-\frac{\alpha_j x_{j-1}}{\varepsilon}\right) - G_j\, exp\left(-\frac{\alpha_j x_j}{\varepsilon}\right)\right] + \frac{G_j - F_j}{H_j}\, exp\left(-\frac{\alpha_j x}{\varepsilon}\right) \\ &+\frac{\gamma_j x}{\alpha_j} - \frac{\gamma_j \varepsilon}{\alpha_j^2} \end{aligned} \tag{6.15}$$

where,

$$H_j = \left[exp\left(-\frac{\alpha_j x_{j-1}}{\varepsilon}\right) - exp\left(-\frac{\alpha_j x_j}{\varepsilon}\right)\right]$$

$$F_j = u_j - \frac{\gamma_j x_j}{\alpha_j} + \frac{\gamma_j \varepsilon}{\alpha_j^2}\ ,\ G_j = u_{j-1} - \frac{\gamma_j x_{j-1}}{\alpha_j} + \frac{\gamma_j \varepsilon}{\alpha_j^2}$$

### 6.3.2 Spline in Tension (ST)

Consider the following singularly perturbed linear two-point bondary value problem:

$$\left.\begin{aligned} Ly &\equiv \varepsilon y'' + a(x)y' + b(x)y = +f(x) \\ y(0) &= \eta_0\ ,\ y(1) = \eta_1\ ;\ \eta_0, \eta_1 \in R \end{aligned}\right\} \tag{6.16}$$

where $a(x) < c < 0$, $b(x) < 0$, $c$ is some constant and $\varepsilon$ is a small positive parameter.

For $x \in [x_{j-1}, x_j]$, we define

$$\alpha_j = \frac{a_{j-1} + a_j}{2}\ ,\ \beta_j = \frac{b_{j-1} + b_j}{2} \text{ and } \gamma_j = \frac{f_{j-1} + f_j}{2}$$

First we consider the case in which $a(x) \equiv 0$.

The approximate solution of the problem (6.16), when $a(x) \equiv 0$, is sought in the form of the function $S(x)$, which on each interval $[x_{j-1}, x_j]$, (denoted by $S_j(x)$) satisfies the following relations :-

(i) the differential equation

$$-\ \varepsilon S_j''(x) + \beta_j S_j(x) = \gamma_j \tag{6.17}$$

(ii) the interpolating conditions

$$S_j(x_{j-1}) = u_{j-1},\ S_j(x_j) = u_j \tag{6.18}$$

(iii) the continuity condition

$$S_j'(x_j^+) = S_j'(x_j^-) \tag{6.19}$$

where,

$$x \in [x_{j-1}, x_j], \;\; x_j = jh, \;\; j = 1(1)n, \;\; h = 1/n.$$

Solving equation (6.17) with the help of (6.18), we obtain

$$S_j(x) = \frac{1}{-sinh\; g_j h}\left[A_j\; sinh\; g_j(x_{j-1} - x) + B_j\; sinh\; g_j(x - x_j)\right] + \frac{\gamma_j}{\beta_j} \tag{6.20}$$

where,

$$A_j = u_j - \frac{\gamma_j}{\beta_j}, \;\; B_j = u_{j-1} - \frac{\gamma_j}{\beta_j}, \;\; g_j = \sqrt{\frac{\beta_j}{\varepsilon}}$$

Equation (6.20) is known as spline in tension [197].

Next, we consider the most general problem, i.e., Equation (6.16) when both $a(x)$ and $b(x)$ are non zero.

In this case the corresponding differential equation will be:

$$-\; \varepsilon S_j''(x) + \alpha_j S_j'(x) + \beta_j S_j(x) = \gamma_j \tag{6.21}$$

Solving (6.21) with the help of (6.18), we obtain

$$\begin{aligned} S_j(x) \;\; &= \;\; \frac{1}{-sinh\; g_j h}\, exp\left(\frac{\alpha_j x}{2\varepsilon}\right)\left[G_j\; sinh\; g_j(x_{j-1} - x) + H_j\; sinh\; g_j(x - x_j)\right] \\ &\quad + \frac{\gamma_j}{\beta_j} \end{aligned} \tag{6.22}$$

where,

$$G_j = \left(u_j - \frac{\gamma_j}{\beta_j}\right) exp\left(-\frac{\alpha_j x_j}{2\varepsilon}\right), \;\; H_j = \left(u_{j-1} - \frac{\gamma_j}{\beta_j}\right) exp\left(-\frac{\alpha_j x_{j-1}}{2\varepsilon}\right)$$

$$g_j = \frac{\sqrt{\alpha_j^2 + 4\beta_j \varepsilon}}{2\varepsilon}$$

### 6.3.3 Variable Mesh Cubic Spline (VMCS)

Consider the following singularly perturbed linear two-point bondary value problem:

$$\left.\begin{array}{l} Ly \;\equiv\; \varepsilon y'' \;+\; a(x)y' \;-\; b(x)y \;=\; f(x) \\ y(0) \;=\; \eta_0 \;\;,\;\; y(1) \;=\; \eta_1 \;\;;\;\; \eta_0, \eta_1 \in R \end{array}\right\} \tag{6.23}$$

where $a(x) > c > 0$, $b(x) > 0$, $c$ is some constant and $\varepsilon$ is a small positive parameter.

The approximate solution of the problem (6.23) is sought in the form of the cubic spline function, which on each interval $[x_{j-1}, x_j]$, denoted by $S_j(x)$ and will be defined as follows:

Let

$$x_0 = 0 \quad x_j = x_0 + \sum_{m=1}^{j} h_m \ , j = 1(1)n, \ h_m = x_m - x_{m-1}, \ x_n = 1$$

For the values $y(x_0), y(x_1), \cdots, y(x_n)$, there exists an interpolating cubic spline with the following properties:

(i) $S_j(x)$ coincides with a polynomial of degree 3 on each interval $[x_{j-1}, x_j], j = 1(1)n$

(ii) $S_j(x) \in C^2[0, 1]$

(iii) $S_j(x_j) = y(x_j), \ j = 0(1)n$

Hence as in [7], the cubic spline can be given as :

$$\begin{aligned} S_j(x) &= \frac{(x_j - x)^3}{6h_j} M_{j-1} + \frac{(x - x_{j-1})^3}{6h_j} M_j + \left( y_{j-1} - \frac{h_j^2 M_{j-1}}{6} \right) \left( \frac{x_j - x}{h_j} \right) \\ &+ \left( y_j - \frac{h_j^2 M_j}{6} \right) \left( \frac{x - x_{j-1}}{h_j} \right) \end{aligned} \tag{6.24}$$

where,

$$x \in [x_{j-1}, x_j] \ , \ h_j = x_j - x_{j-1} \ , \ j = 1(1)n$$

and

$$M_j = S_j''(x_j) \ , \ j = 0(1)n$$

# 6.4 Derivation of the schemes

## 6.4.1 Derivation of the scheme using Spline in Compression

Since $S(x) \in C^2[0, 1]$, therefore we have

$$S_j'(x_j) = S_{j+1}'(x_j) \tag{6.25}$$

Differentiating (6.11) with respect to $x$, putting $x = x_j$ and using (6.25), we obtain the difference scheme

$$Ru_j = Qf_j \ , \ j = 1(1)n - 1 \tag{6.26}$$

where,

$$Ru_j = r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1}$$

$$Qf_j = q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1}$$

$$u_0 = \eta_0 \ , \ u_n = \eta_1$$

$$\left.\begin{array}{c} r_j^- = \left(1 - \frac{t_j^2}{4}\right)\left(\frac{2-p_j}{2+p_j}\right) \ , \ r_j^+ = \left(1 - \frac{t_{j+1}^2}{4}\right)\left(\frac{2+p_{j+1}}{2-p_{j+1}}\right) \\ r_j^c = -2 + p_j - p_{j+1} - \frac{1}{4}(t_j^2 + t_{j+1}^2) \\ q_j^- = \frac{h^2}{2\varepsilon(2+p_j)} \ , \ q_j^+ = \frac{h^2}{2\varepsilon(2-p_{j+1})} \ , \ q_j^c = q_j^- + q_j^+ \end{array}\right\} \quad (6.27)$$

**Remark 2**: The coefficients $r's$ and $q's$ in (6.26) for (6.12), which can be obtained using (6.15) and (6.25), will be given by

$$\left.\begin{array}{c} r_j^- = 1 - \frac{p_j}{2}, \ r_j^+ = 1 + \frac{p_{j+1}}{2}, \ r_j^c = -(r_j^- + r_j^+) \\ q_j^- = q_j^+ = \frac{h^2}{4\varepsilon}, \ q_j^c = \frac{h^2}{2\varepsilon} \end{array}\right\} \quad (6.28)$$

### 6.4.2 Derivation of the scheme using Spline in Tension

Since $S(x) \in C^2[0,1]$, therefore we have

$$S_j'(x_j) = S_{j+1}'(x_j) \quad (6.29)$$

Differentiating (6.20) with respect to $x$, putting $x = x_j$ and using (6.29), we obtain the difference scheme

$$Ru_j = Qf_j \ , \ j = 1(1)n-1 \quad (6.30)$$

where,

$$Ru_j = r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1}$$

$$Qf_j = q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1}$$

$$u_0 = \eta_0 \ , \ u_n = \eta_1$$

$$\left.\begin{array}{c} r_j^- = \frac{p_j}{sinh\, p_j} \ , \ r_j^+ = \frac{p_{j+1}}{sinh\, p_{j+1}} \ , \ r_j^c = -(p_j\, coth\, p_j + p_{j+1}\, coth\, p_{j+1}) \\ q_j^- = -\frac{p_j}{2\beta_j}\left(coth\, p_j - \frac{1}{sinh\, p_j}\right) \ , \ q_j^+ = -\frac{p_{j+1}}{2\beta_{j+1}}\left(coth\, p_{j+1} - \frac{1}{sinh\, p_{j+1}}\right) \\ q_j^c = q_j^- + q_j^+ \end{array}\right\} \quad (6.31)$$

**Remark 1**: The coefficients $r's$ and $q's$ in (6.30) for (6.21), which can be obtained using (6.22) and (6.29), will be given by

$$\left.\begin{array}{c} r_j^- = \left(\frac{p_j}{sinh\, p_j}\right) exp(t_j)\ ,\ r_j^+ = \left(\frac{p_{j+1}}{sinh\, p_{j+1}}\right) exp(-t_{j+1}) \\ r_j^c = -p_j\, coth\, p_j - p_{j+1}\, coth\, p_{j+1} + t_{j+1} - t_j \\ q_j^- = \frac{1}{2\beta_j}(r_j^- - t_j - p_j\, coth\, p_j)\ ,\ q_j^+ = \frac{1}{2\beta_{j+1}}(r_j^+ + t_{j+1} - p_{j+1}\, coth\, p_{j+1}) \\ q_j^c = q_j^- + q_j^+ \end{array}\right\} \quad (6.32)$$

## 6.4.3 Derivation of the scheme using Cubic Spline

Differentiating (6.24) and denoting the approximate solution to $y(x)$ by $u(x)$, we get

$$S_j'(x) = -\frac{(x_j - x)^2}{2h_j} M_{j-1} + \frac{(x - x_{j-1})^2}{2h_j} M_j + \left(\frac{u_j - u_{j-1}}{h_j}\right) - \left(\frac{M_j - M_{j-1}}{6}\right) h_j \quad (6.33)$$

Since $S_j(x) \in C^2[0,1]$, therefore we have

$$S_j'(x_j) = S_{j+1}'(x_j) \quad (6.34)$$

$$\Rightarrow \quad \frac{h_j}{6} M_{j-1} + \frac{h_j + h_{j+1}}{6} M_j + \frac{h_{j+1}}{6} M_{j+1} = \frac{u_{j+1} - u_j}{h_{j+1}} - \frac{u_j - u_{j-1}}{h_j} \quad (6.35)$$

where

$$M_j = \frac{1}{\varepsilon}\left(f_j - a_j u_j' + b_j u_j\right) \quad (6.36)$$

Taking the Taylor series expansion for $u$ around $x_j$ and neglecting the terms containing third and higher order terms, we get the following approximations for $u_{j+1}$ and $u_{j-1}$ :

$$u_{j+1} \approx u_j + h_{j+1} u_j' + \frac{h_{j+1}^2}{2} u_j'' \quad (6.37)$$

$$u_{j-1} \approx u_j - h_j u_j' + \frac{h_j^2}{2} u_j'' \quad (6.38)$$

From equations (6.37) and (6.38), we get

$$u_j' = \frac{1}{h_j h_{j+1}(h_j + h_{j+1})}\left[h_j^2 u_{j+1} - \left(h_j^2 - h_{j+1}^2\right) u_j - h_{j+1}^2 u_{j-1}\right] \quad (6.39)$$

and

$$u_j'' = \frac{2}{h_j h_{j+1}\left(h_j + h_{j+1}\right)}\left[h_j u_{j+1} - \left(h_j + h_{j+1}\right) u_j - h_{j+1} u_{j-1}\right] \tag{6.40}$$

Also

$$u_{j+1}' \approx u_j' + h_{j+1} u_j'' \tag{6.41}$$

and

$$u_{j-1}' \approx u_j' - h_j u_j'' \tag{6.42}$$

From equations (6.39), (6.40) and (6.41), we get

$$u_{j+1}' \approx \frac{1}{h_j h_{j+1}\left(h_j + h_{j+1}\right)}\left[h_{j+1}^2 u_{j-1} - \left(h_j + h_{j+1}\right)^2 u_j + \left(h_j^2 + 2h_j h_{j+1}\right) u_{j+1}\right] \tag{6.43}$$

and from equations (6.39), (6.40) and (6.42), we get

$$u_{j-1}' \approx \frac{1}{h_j h_{j+1}\left(h_j + h_{j+1}\right)}\left[-\left(h_{j+1}^2 + 2h_j h_{j+1}\right) u_{j-1} - \left(h_j + h_{j+1}\right)^2 u_j - h_j^2 u_{j+1}\right] \tag{6.44}$$

Therefore, using (6.35), (6.36), (6.39), (6.43) and (6.44), we obtain the difference scheme

$$Ru_j = QZ_j \ , \ j = 1(1)n-1 \tag{6.45}$$

where,

$$Ru_j = r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1}$$

$$QZ_j = q_j^- Z_{j-1} + q_j^c Z_j + q_j^+ Z_{j+1}$$

$$u_0 = \alpha_0 \ , \ u_n = \alpha_1$$

$$r_j^- = \frac{2h_j + h_{j+1}}{6(h_j + h_{j+1})} a_{j-1} + \frac{h_{j+1}}{3h_j} a_j - \frac{h_{j+1}^2}{6h_j(h_j + h_{j+1})} a_{j+1} + \frac{h_j}{6} b_{j-1} - \frac{\varepsilon}{h_j}$$

$$r_j^+ = \frac{h_j^2}{6h_{j+1}(h_j + h_{j+1})} a_{j-1} - \frac{h_j}{3h_{j+1}} a_j - \frac{2h_{j+1} + h_j}{6(h_j + h_{j+1})} a_{j+1} + \frac{h_{j+1}}{6} b_{j+1} - \frac{\varepsilon}{h_{j+1}}$$

$$r_j^c = -\frac{h_j + h_{j+1}}{6h_{j+1}} a_{j-1} - \frac{h_{j+1}^2 - h_j^2}{3h_j h_{j+1}} a_j + \frac{h_j + h_{j+1}}{6h_j} a_{j+1} + \frac{h_j + h_{j+1}}{3} b_j + \frac{\varepsilon}{h_j} + \frac{\varepsilon}{h_{j+1}}$$

$$q_j^- = -\frac{h_j}{6}, \ q_j^+ = -\frac{h_{j+1}}{6}, \ q_j^c = -\frac{h_j + h_{j+1}}{3} \tag{6.46}$$

## Mesh Selection Strategy

We form the non-uniform grid in such a way that more points are generated in the boundary layer regions than outside these regions.

Let the concerned interval on which the problem is to be solved is $[p_1, p_2]$, where $p_1 = 0$ and $p_2 > 0$. Let $\delta$ denotes the width of the boundary layer.

We have two cases:-

Case I : There are two boundary layers (one at both ends).

Case II : There is one boundary layer at the left end.

First we consider the case of two boundary layers.

In this case we have three subintervals:

$$[p_1\,,\,p_1+\delta] \quad , \quad [p_1+\delta\,,\,p_2-\delta] \quad \text{and} \quad [p_2-\delta\,,\,p_2]$$

Let $n_1$, $n_2$ and $n_3$ be the number of points in these subintervals, respectively, such that $n_1 + n_2 + n_3 = n$ and $n_1 = n_3$. Further let the positive constants $\widetilde{h}_1$ and $K$ be known. Then we generate the mesh as follows:

On the interval $[p_1\,,\,p_1+\delta]$, the grid is non-uniform and is defined as follows:

$\widetilde{h}_j = \widetilde{h}_{j-1} + K\left[\frac{\widetilde{h}_{j-1}}{\varepsilon}\right]\min\left(\widetilde{h}^2_{j-1}, \varepsilon\right) \quad , \quad j = 2(1)n_1.$

Now, let

$\widetilde{q} = \sum_{j=1}^{n_1} \widetilde{h}_j$

$q = \frac{\delta}{\widetilde{q}}$

$h_j = q\widetilde{h}_j \quad , \quad j = 1(1)n_1$

On the interval $[p_1+\delta\,,\,p_2-\delta]$, the grid is uniform and is defined as follows :

$h_j = \frac{(p_2-p_1-2\delta)}{n_2} \quad , \quad j = n_1+1\,, n_1+n_2.$

On the interval $[p_2-\delta\,,\,p_2]$ the grid is the mirror image of the grid on $[p_1\,,\,p_1+\delta]$ and

therefore will be given by:

$h_j = h_{n+1-j} \quad , \quad j = n_1 + n_2 + 1 \, , \, n.$

and define

$x_0 = p_1$

$x_j = x_{j-1} + h_j \quad , \quad j = 1(1)n.$

Now we consider the second case in which there is one boundary layer at the left end of the underlying interval.

In this case we have two subintervals, viz.,

$$[p_1 \, , \, p_1 + \delta] \text{ and } [p_1 + \delta \, , \, p_2]$$

Let $n_1$ and $n_2$ be the number of points in these two subintervals, respectively, such that $n_1 + n_2 = n$. We define non-uniform mesh in the interval $[p_1 \, , \, p_1 + \delta]$ in a similar manner as we did in the first case and then in the subinterval $[p_1 + \delta \, , \, p_2]$ we define uniform mesh which will be given by

$h_j = \frac{(p_2 - p_1 - \delta)}{n_2} \quad , \quad j = n_1 + 1 \, , \, n.$

## 6.5 Proof of the uniform convergence

We will prove first the convergence of the sequence $\{y_n\}$ and then we will give the error estimates for the linear problems.

Convergence:-

For the sake of convenience, throughout in the convergence part we denote $f(x, y, y')$ by $f(y)$.

Consider

$$\left.\begin{array}{c} \varepsilon y'' = f(y) \\ y(x_0) = 0 \, , \, y(x_n) = 0 \end{array}\right\} \tag{6.47}$$

Let $y_0(x)$ be some initial approximation and consider the sequence $\{y_n\}$ determined by the recurrence relation

$$\left.\begin{array}{c} \varepsilon y_n'' = f(y_{n-1}) + (y_n - y_{n-1}) f'(y_{n-1}) \\ y_n(x_0) = 0 \, , \, y_n(x_n) = 0 \end{array}\right\} \tag{6.48}$$

$$\Rightarrow \varepsilon(y_{n+1}-y_n)'' = f(y_n) - f(y_{n-1}) - (y_n - y_{n-1})f'(y_{n-1}) + (y_{n+1}-y_n)f'(y_n)$$

Regarding this as a differential equation for $(y_{n+1} - y_n)$ and converting into an integral equation, we have

$$\begin{aligned}\varepsilon(y_{n+1}-y_n) &= \int_{x_0}^{x_n} G(x,s)\{f(y_n) - f(y_{n-1}) - (y_n - y_{n-1})f'(y_{n-1}) \\ &\quad +(y_{n+1}-y_n)f'(y_n)\}dy\end{aligned}$$

where, the Green's function $G(x,s)$ will be given by

$$G(x,s) = \begin{cases} \frac{(x_n-x)(s-x_0)}{(x_n-x_0)} & , \quad x_0 \le s \le x \\ \frac{(x-x_0)(x_n-s)}{(x_n-x_0)} & , \quad x \le s \le x_n \end{cases}$$

$G(x,s)$ assumes its maximum value of $(x_n - x_0)/4$ at $s = (x_0 + x_n)/2$.

By Mean Value Theorem, we have

$$f(y_n) - f(y_{n-1}) - (y_n - y_{n-1})f'(y_{n-1}) = \frac{(y_n - y_{n-1})^2}{2} f''(\theta) \quad : \quad y_{n-1} < \theta < y_n$$

and let

$$\max_{|y|\le 1} |f'(y)| = m \quad , \quad \max_{|y|\le 1} |f''(y)| = k$$

Thus

$$\max_x |y_{n+1} - y_n| \le \frac{x_n - x_0}{4} \int_{x_0}^{x_n} \left[ \max_x \frac{(y_n - y_{n-1})^2}{2} k + \max_x |y_{n+1} - y_n| m \right] dy$$

A simple calculation yields

$$\max_x |y_{n+1} - y_n| \le \left[ \frac{k(x_n - x_0)^2/8\varepsilon}{1 - m(x_n - x_0)^2/4\varepsilon} \right] \left( \max_x (|y_n - y_{n-1}|)^2 \right)$$

This shows that there is a quadratic convergence if $\frac{k(x_n-x_0)^2/8\varepsilon}{1-m(x_n-x_0)^2/4\varepsilon} < 1$. This quantity can be shown to be less than one for $(x_n - x_0)$ sufficiently small. If $(x_n - x_0)$ is too large initially, still we can keep $|y_1(x) - y_0(x)|$ sufficiently small by choosing the judicious initial approximation $y_0(x)$. This will retain $\max |y_{n+1} - y_n|$ small enough for all $x \in (x_0, x_n)$, which is sufficient for convergence.

Error Estimates for The Linear Problems:-

As we mentioned in the introduction of this chapter the first two methods SC and ST are the generalizations of the methods developed for linear problems. So in this part we are giving the error estimate for the third method (VMCS) only.

Using the third method (VMCS) we have solved two types of problems, viz., the problems when there are two boundary layers (one at both ends) and the problems in which there is only one boundary layer at the left end.

For the error analysis of the problems having twin boundary layers, we have used the comparison functions method developed by Kellogg and Tsan [131] and Berger et al. [26]. Analysis for the case of only one boundary layer can be done analogously (see, e.g., Berger et al. [26], for details).

For the sake of simplicity, we consider $p_1 = 0$ and $p_2 = 1$. Also throughout the paper $M$ will denote positive constant which may take different values in different equations(inequalities) but that are always independent of $h$ and $\varepsilon$.

This method uses the following two Lemmas [26]:

**Lemma 6.1** *(maximum principle) : Let $\{u_j\}$ be a set of values at the grid points $x_j$, satisfying $u_0 \leq 0, u_n \leq 0$ ; $(u_0 = u(x = 0))$ and $Ru_j \geq 0, j = 1(1)n-1$, then $u_j \leq 0, j = 0(1)n$.*

**Lemma 6.2** *If $K_1(h,\varepsilon) \geq 0$ and $K_2(h,\varepsilon) \geq 0$ are such that*

$$R(K_1(h,\varepsilon)\phi_j + K_2(h,\varepsilon)\psi_j) \geq R(\pm e_j) = \pm\tau_j(y)$$

*for each $j = 1, 2, \cdots, n-1$, then the discrete maximum principle implies that*

$$|e_j| \leq K_1(h,\varepsilon)|\phi_j| + K_2(h,\varepsilon)|\psi_j|$$

*where, $|e_j| = |y(x_j) - u_j|$, for each $j$ and $\phi$ and $\psi$ are two comparison functions.*

The following Lemma (which is a slite extension of the Lemma 2.4 in [131]), gives the properties of the exact solution of (6.23):

**Lemma 6.3** *If $y$ satisfies (6.23), then*

$$y(x) = v(x) + w(x) + g(x)$$

*where,*

$$v(x) = \left(-\frac{\varepsilon y'(0)}{a(0)}\right) exp\left(-\frac{a(0)}{\varepsilon}(x)\right)$$

$$w(x) = \left(-\frac{\varepsilon y'(1)}{a(1)}\right) exp\left(-\frac{a(1)}{\varepsilon}(1-x)\right)$$

*and*

$$\left|g^{(k)}(x)\right| \leq M\left\{1 + \varepsilon^{-k+1} exp\left(-\frac{c}{\varepsilon}(1-x)\right)\right\}$$

$k = 0(1)4$, *c is some constant and* $M$ *is a positive constant independent of* $h$ *and* $\varepsilon$.

We use the following two comparison functions (as in [27]):

$$\phi = C_1 exp\left[-2\eta C_3 \frac{x}{\varepsilon}\right] \quad \text{and} \quad \psi = C_2 exp\left[-2\eta C_3 \frac{1-x}{\varepsilon}\right]$$

where, $C_1$, $C_2$ and $\eta$ are constants independent of $h$ and $\varepsilon$.

Remark :- The following inequalities hold:

$$R\phi_j \geq M \quad , \quad h_c^2 \leq \varepsilon$$

$$R\phi_j \geq M h_c \ , \quad h_c^2 \geq \varepsilon$$

$$R\psi_j \geq M \quad , \quad h_c^2 \leq \varepsilon$$

$$R\psi_j \geq M h_c \ , \quad h_c^2 \geq \varepsilon$$

where

$$h_c = \max_j h_j \ ( = \text{a constant})$$

Now we estimate the truncation error of the scheme (6.45) using (6.46).

First consider the case in which $h_c^2 \leq \varepsilon$.

We have,

$$\tau_j(y) = T_0 y_j + T_1 y_j' + T_2 y_j'' + T_3 y_j''' + \text{Remainder terms}$$

where,

$$\begin{aligned}
T_0 &= \left(r_j^- + r_j^c + r_j^+\right) + \left(q_j^- b_{j-1} + q_j^c b_j + q_j^+ b_{j+1}\right) \\
T_1 &= \left(h_{j+1} r_j^+ - h_j r_j^-\right) - \left\{q_j^-\left(a_{j-1} + h_j b_{j-1}\right) + q_j^c a_j + q_j^+\left(a_{j+1} - h_{j+1} b_{j+1}\right)\right\} \\
T_2 &= \left(\frac{h_j^2}{2} r_j^- + \frac{h_{j+1}^2}{2} r_j^+\right) - \varepsilon\left(q_j^- + q_j^c + q_j^+\right) \\
&\quad + \left\{q_j^-\left(h_j a_{j-1} + \frac{h_j^2}{2} b_{j-1}\right) + q_j^+\left(-h_{j+1} a_{j+1} + \frac{h_{j+1}^2}{2} b_{j+1}\right)\right\}
\end{aligned}$$

$$T_3 = \left(\frac{h_{j+1}^3}{6}r_j^+ - \frac{h_j^3}{6}r_j^-\right) + \varepsilon\left(q_j^- h_j - q_j^+ h_{j+1}\right) - q_j^-\left\{\frac{h_j^2}{2}a_{j-1} + \frac{h_j^3}{6}b_{j-1}\right\}$$
$$- q_j^+\left\{\frac{h_{j+1}^2}{2}a_{j+1} - \frac{h_{j+1}^3}{6}b_{j+1}\right\}$$

Using (6.46) we see that $T_0 = 0, T_1 = 0, T_2 = 0$ and $|T_3| \le Mh_c^3$

Now from Lemma 6.3, we have

$$v_j''' = \left(-\frac{a(0)}{\varepsilon}\right)^2 y'(0)\, exp\left\{-\frac{a(0)x_j}{\varepsilon}\right\}$$

therefore

$$|\tau_j(v)| \le \frac{Mh_c^3}{\varepsilon^2}\, exp\left\{-\frac{a(0)x_j}{\varepsilon}\right\} \quad , \quad h_c^2 \le \varepsilon$$

Similarly,

$$|\tau_j(w)| \le M\frac{h_c^3}{\varepsilon^2}\, exp\left\{-\frac{a(1)(1-x_j)}{\varepsilon}\right\} \quad , \quad h_c^2 \le \varepsilon$$

Also

$$\left|g_j^{(3)}\right| \le M\left[1 + \varepsilon^{-2}\, exp\left\{-\frac{c(1-x_j)}{\varepsilon}\right\}\right]$$
$$\Rightarrow |\tau_j(g)| \le Mh_c^3\left[1 + \frac{1}{\varepsilon^2}\, exp\left\{-\frac{c(1-x_j)}{\varepsilon}\right\}\right] \quad , \quad h_c^2 \le \varepsilon$$

Now

$$\tau_j(y) = \tau_j(v) + \tau_j(w) + \tau_j(g)$$
$$\Rightarrow \quad |\tau_j(y)| \le M\frac{h_c^3}{\varepsilon^2}\left[1 + exp\left\{-\frac{a(0)x_j}{\varepsilon}\right\} + exp\left\{-\frac{a(1)(1-x_j)}{\varepsilon}\right\}\right] \quad , \quad h_c^2 \le \varepsilon$$

In the opposite case, i.e., when $h_c^2 \ge \varepsilon$, we use the following expression for truncation error:

$$\tau_j(y) = \left(\frac{h_{j+1}^3}{6}r_j^+ - \frac{h_j^3}{6}r_j^-\right) y'''(\xi_1) + \varepsilon\left(q_j^- h_j - q_j^+ h_{j+1}\right) y'''(\xi_2)$$
$$-q_j^-\left\{\frac{h_j^2}{2}a_{j-1} + \frac{h_j^3}{6}b_{j-1}\right\} y'''(\xi_3) - q_j^+\left\{\frac{h_{j+1}^2}{2}a_{j+1} - \frac{h_{j+1}^3}{6}b_{j+1}\right\} y'''(\xi_4)$$
$$x_{j-1} < \xi_i < x_{j+1} \, , \; i = 1(1)4$$

Now

$$\left|\frac{h_{j+1}^3}{6}r_j^+ - \frac{h_j^3}{6}r_j^-\right| \le Mh_c^3 \quad , \quad \left|\varepsilon\left(q_j^- h_j - q_j^+ h_{j+1}\right)\right| \le Mh_c^3$$
$$\left|q_j^-\left(\frac{h_j^2}{2}a_{j-1} + \frac{h_j^3}{6}b_{j-1}\right)\right| \le Mh_c^3 \quad , \quad \left|q_j^+\left(\frac{h_{j+1}^2}{2}a_{j+1} - \frac{h_{j+1}^3}{6}b_{j+1}\right)\right| \le Mh_c^3$$

Using these estimates and the above expression for $\tau_j(y)$, we obtain the same estimates for $\tau_j(v)$, $\tau_j(w)$ and $\tau_j(g)$ as were in the case of $h_c^2 \leq \varepsilon$. Choosing

$$K_1 = h_c^2 exp\left\{-\frac{a(0)x_j}{\varepsilon}\right\} \quad \text{and} \quad K_2 = h_c^2 exp\left\{-\frac{a(1)(1-x_j)}{\varepsilon}\right\}$$

we see that the Lemma 6.2 is satisfied(in both the cases $h_c^2 \leq \varepsilon$ and $h_c^2 \geq \varepsilon$) and thus we have proved the following theorem:

**Theorem 6.1** *Let $\{u_j\}$, $j = 0(1)n$, be a set of values of the approximate solution to $y(x)$ of (6.23), obtained using (6.45) and (6.46). Then there are positive constants $\widetilde{C}_1$, $\widetilde{C}_2$ and $M$ (independent of $h$ and $\varepsilon$) such that the following estimate holds :*

$$\max_j |y_j - u_j| \leq M h_c^2 \left[ exp\left\{-\frac{\widetilde{C}_1 x_j}{\varepsilon}\right\} + exp\left\{-\frac{\widetilde{C}_2(1-x_j)}{\varepsilon}\right\}\right]$$

where

$$h_c = \max_j h_j = \text{a constant.}$$

## 6.6 Test Examples and Numerical Results

To illustrate the predicted theory, we solve the following singularly perturbed non-linear problems:

**Example 6.1** [24] : *Consider* $\varepsilon y'' + 2y' + e^y = 0$ ; $y(0) = 0$ , $y(1) = 0$

*Exact solution : Not available.*

*Quasilinear process equations are*

$$\varepsilon(y'')^{(k+1)}(x) + 2(y')^{(k+1)}(x) + e^{y^{(k)}(x)} y^{(k+1)}(x) = e^{y^{(k)}(x)}\left(y^{(k)}(x) - 1\right)$$

$$y^{(k)}(0) = 0 \ , \ y^{(k)}(1) = 0$$

*Method used : (6.26) together with (6.27).*

**Example 6.2** [176] : *Consider* $\varepsilon y'' + (2x+1)y' + y^2 = 0$ ; $y(0) = 1$ , $y(1) = 1$

*Exact solution : Not available.*

*Quasilinear process equations are*

$$\varepsilon(y'')^{(k+1)}(x) + (2x+1)(y')^{(k+1)}(x) + 2y^{(k)}(x)y^{(k+1)}(x) = \left(y^{(k)}(x)\right)^2$$

$$y^{(k)}(0) = 1 \ , \ y^{(k)}(1) = 1$$

*Method used : (6.26) together with (6.27).*

**Example 6.3** *[49] : Consider* $\varepsilon y'' + (y')^2 + yy' = 0 \ ; \ y(0) = 2 \ , \ y(1) = 1$

*Exact solution : Not available.*

*Quasilinear process equations are*

$$\varepsilon(y'')^{(k+1)}(x) + \left(2(y')^{(k)}(x)y^{(k)}(x)\right)(y')^{(k+1)}(x) + (y')^{(k)}(x)y^{(k+1)}(x)$$
$$= \left(y^{(k)}(x)\right)^2 + \left((y')^{(k)}(x)y^{(k)}(x)\right)$$

$$y^{(k)}(0) = 2 \ , \ y^{(k)}(1) = 1$$

*Method used : (6.26) together with (6.27).*

**Example 6.4** *[190] : Consider* $\varepsilon y'' + (y')^2 = 1 \ ; \ y(0) = 1 \ , \ y(1) = 1$

*Exact solution is given by*

$$y(x) = 1 - x + 2\varepsilon\left[log\left(\sqrt{1 + e^{(2x-1)/\varepsilon}}\right) - log\left(\sqrt{1 + e^{-1/\varepsilon}}\right)\right]$$

*Quasilinear process equations are*

$$\varepsilon(y'')^{(k+1)}(x) + 2(y')^{(k)}(x)(y')^{(k+1)}(x) = 1 + \left(y^{(k)}(x)\right)^2$$

$$y^{(k)}(0) = 1 \ , \ y^{(k)}(1) = 1$$

*Method used : (6.26) together with (6.28).*

**Example 6.5** *[24] : Consider* $\varepsilon y'' - xy - y^2 = 0 \ ; \ y(0) = 1 \ , \ y(1) = 0$

*Exact solution : Not available.*

*Quasilinear process equations are*

$$\varepsilon(y'')^{(k+1)}(x) + \left(-x - 2y^{(k)}(x)\right)y^{(k+1)}(x) = -\left(y^{(k)}(x)\right)^2$$

$$y^{(k)}(0) = 1 \ , \ y^{(k)}(1) = 0$$

*Method used : (6.30) together with (6.31).*

**Example 6.6** [46] : *Consider* $\varepsilon y'' - y - y^2 = -e^{-2x/\sqrt{\varepsilon}}$ ; $y(0) = 1$ , $y(1) = e^{-1/\sqrt{\varepsilon}}$

*Exact solution is given by*

$$y(x) = e^{-x/\sqrt{\varepsilon}}$$

*Quasilinear process equations are*

$$\varepsilon(y'')^{(k+1)}(x) + \left(-1 - 2y^{(k)}(x)\right) y^{(k+1)}(x) = -\left(y^{(k)}(x)\right)^2 - e^{-2x/\sqrt{\varepsilon}}$$

$$y^{(k)}(0) = 1 \ , \ y^{(k)}(1) = e^{-1/\sqrt{\varepsilon}}$$

*Method used : (6.30) together with (6.31).*

**Example 6.7** [176] : *Consider* $\varepsilon y'' - yy' - y = 0$ ; $y(0) = 1$ , $y(1) = 1$

*Exact solution : Not available.*

*Quasilinear process equations are*

$$\varepsilon(y'')^{(k+1)}(x) + \left(-y^{(k)}(x)(y')^{(k+1)}(x)\right) + \left(-1 - (y')^{(k)}(x)\right) y^{(k+1)}(x) = -(y')^{(k)}(x)y^{(k)}(x)$$

$$y^{(k)}(0) = 1 \ , \ y^{(k)}(1) = 1$$

*Method used : (6.30) together with (6.32).*

**Example 6.8** [93] : *Consider* $\varepsilon y'' + yy' - y = 0$ ; $y(0) = 1$ , $y(1) = -\frac{1}{3}$

*Exact solution : Not available.*

*Quasilinear process equations are*

$$\varepsilon(y'')^{(k+1)}(x) + y^{(k)}(x)(y')^{(k+1)}(x) + \left(-1 + (y')^{(k)}(x)\right) y^{(k+1)}(x) = y^{(k)}(x)(y')^{(k)}(x)$$

$$y^{(k)}(0) = 1 \ , \ y^{(k)}(1) = -\frac{1}{3}$$

**Example 6.9** [46] : *Consider* $\varepsilon y'' - y - y^2 = -e^{-2x/\sqrt{\varepsilon}}$ ; $y(0) = 1$ , $y(1) = e^{-1/\sqrt{\varepsilon}}$
*Exact solution is given by*

$$y(x) = e^{-x/\sqrt{\varepsilon}}$$

*Quasilinear process equations are*

$$\varepsilon (y'')^{(k+1)}(x) + \left(-1 - 2y^{(k)}(x)\right) y^{(k+1)}(x) = -\left(y^{(k)}(x)\right)^2 - e^{-2x/\sqrt{\varepsilon}}$$

$$y^{(k)}(0) = 1 \ , \ \ y^{(k)}(1) = e^{-1/\sqrt{\varepsilon}}$$

Tables 6.1, 6.3, 6.5, 6.9, 6.13 and 6.15 - 6.17 contain the maximum errors:

$$\max_{0 \leq j \leq n} \left| u_j^n - u_{2j}^{2n} \right|$$

where $u(x_j)$ is the approximate solution of the corresponding nonlinear problem.

For Examples 6.4, 6.6 and 6.9, we used the following criteria to estimate maximum error:

$$\max_j |y(x_j) - u_j|$$

for different $n$ and $\varepsilon$, $n = 1/h$. The results are presented in Tables 6.7 and 6.11, respectively.

Tables 6.18 and 6.22 contain the numerical rate of uniform convergence:

$$r_{k,\varepsilon} = log_2 \left( z_{k,\varepsilon} / z_{k+1,\varepsilon} \right) \quad \text{where} \quad z_{k,\varepsilon} = \max_j |u_j^{h_j/2^k} - u_{2j}^{h_j/2^{k+1}}| \ , \ \ k = 0, 1, 2, \cdots\cdots\cdots$$

and $u_j^{h_j/2^k}$ denotes the value of $u_j$ for the mesh length $h_j/2^k$.

(The other tables, viz., Tables 6.2, 6.4, 6.6, 6.8, 6.10, 6.12 and 6.14 also contain the numerical rate of uniform convergence but in these tables results are tabulated for $h_j = h =$ constant).

Table 6.1: Max. Errors for Example 6.1

| $\varepsilon$ | n = 64 | n = 128 | n = 256 | n = 512 | n = 1024 | n = 2048 |
|---|---|---|---|---|---|---|
| 1/10 | 0.34E-02 | 0.84E-03 | 0.21E-03 | 0.53E-04 | 0.13E-04 | 0.33E-05 |
| 1/20 | 0.16E-01 | 0.39E-02 | 0.97E-03 | 0.24E-03 | 0.61E-04 | 0.15E-04 |
| 1/40 | 0.80E-01 | 0.17E-01 | 0.42E-02 | 0.10E-02 | 0.26E-03 | 0.65E-04 |
| 1/80 | — | 0.84E-01 | 0.18E-01 | 0.44E-02 | 0.11E-02 | 0.27E-03 |
| 1/160 | — | — | 0.87E-01 | 0.19E-01 | 0.45E-02 | 0.11E-02 |
| 1/320 | — | — | — | 0.88E-01 | 0.19E-01 | 0.46E-02 |
| 1/640 | — | — | — | — | 0.89E-01 | 0.19E-01 |

Table 6.2: Rate of convergence for Example 6.1
$n = 64, 128, 256, 512, 1024$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/4 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/8 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/16 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/32 | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/64 | 0.26E+01 | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.22E+01 |
| 1/128 | — | 0.27E+01 | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.22E+01 |
| 1/256 | — | — | 0.27E+01 | 0.21E+01 | 0.20E+01 | 0.23E+01 |

Table 6.3: Max. Errors for Example 6.2

| $\varepsilon$ | n = 64 | n = 128 | n = 256 | n = 512 | n = 1024 | n = 2048 |
|---|---|---|---|---|---|---|
| 1/10 | 0.55E-02 | 0.14E-02 | 0.34E-03 | 0.85E-04 | 0.21E-04 | 0.53E-05 |
| 1/20 | 0.31E-01 | 0.73E-02 | 0.18E-02 | 0.45E-03 | 0.11E-03 | 0.28E-04 |
| 1/40 | — | 0.37E-01 | 0.87E-02 | 0.21E-02 | 0.53E-03 | 0.13E-03 |
| 1/80 | — | — | 0.41E-01 | 0.97E-02 | 0.24E-02 | 0.59E-03 |
| 1/160 | — | — | — | 0.44E-01 | 0.10E-01 | 0.26E-02 |
| 1/320 | — | — | — | — | 0.46E-01 | 0.11E-01 |
| 1/640 | — | — | — | — | — | 0.47E-01 |

Table 6.4: Rate of convergence for Example 6.2
$n = 64, 128, 256, 512, 1024$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/16 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/32 | 0.22E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.21E+01 |
| 1/64 | 0.33E+01 | 0.22E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.23E+01 |
| 1/128 | 0.96E+00 | 0.33E+01 | 0.22E+01 | 0.20E+01 | 0.20E+01 | 0.21E+01 |
| 1/256 | — | 0.94E+00 | 0.33E+01 | 0.22E+01 | 0.21E+01 | 0.21E+01 |
| 1/512 | — | — | 0.90E+00 | 0.33E+01 | 0.22E+01 | 0.21E+01 |
| 1/1024 | — | — | — | 0.88E+00 | 0.33E+01 | 0.21E+01 |

Table 6.5: Max. Errors for Example 6.3

| $\varepsilon$ | n = 16 | n = 32 | n = 64 | n = 128 | n = 256 | n = 512 |
|---|---|---|---|---|---|---|
| 1/2 | 0.40E-03 | 0.99E-04 | 0.25E-04 | 0.62E-05 | 0.16E-05 | 0.39E-06 |
| 1/4 | 0.27E-02 | 0.65E-03 | 0.16E-03 | 0.40E-04 | 0.10E-04 | 0.25E-05 |
| 1/8 | 0.18E-01 | 0.37E-02 | 0.88E-03 | 0.22E-03 | 0.54E-04 | 0.13E-04 |
| 1/16 | — | — | 0.43E-02 | 0.10E-02 | 0.25E-03 | 0.63E-04 |
| 1/32 | — | — | — | — | 0.11E-02 | 0.27E-03 |

Table 6.6: Rate of convergence for Example 6.3
$n = 64, 128, 256, 512, 1024$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/16 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/32 | 0.22E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.21E+01 |
| 1/64 | 0.33E+01 | 0.22E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.23E+01 |
| 1/128 | 0.97E+00 | 0.33E+01 | 0.22E+01 | 0.20E+01 | 0.20E+01 | 0.21E+01 |
| 1/256 | — | 0.94E+00 | 0.33E+01 | 0.22E+01 | 0.21E+01 | 0.21E+01 |
| 1/512 | — | — | 0.91E+00 | 0.33E+01 | 0.22E+01 | 0.21E+01 |

Table 6.7: Max. Errors for Example 6.4

| $\varepsilon$ | n = 64 | n = 128 | n = 256 | n = 512 | n = 1024 | n = 2048 |
|---|---|---|---|---|---|---|
| 1/10 | 0.36E-03 | 0.90E-04 | 0.23E-04 | 0.56E-05 | 0.14E-05 | 0.35E-06 |
| 1/20 | 0.72E-03 | 0.18E-03 | 0.45E-04 | 0.11E-04 | 0.28E-05 | 0.70E-06 |
| 1/40 | 0.14E-02 | 0.36E-03 | 0.90E-04 | 0.23E-04 | 0.56E-05 | 0.14E-05 |
| 1/80 | 0.25E-02 | 0.70E-03 | 0.18E-03 | 0.45E-04 | 0.11E-04 | 0.28E-05 |
| 1/160 | 0.38E-02 | 0.12E-02 | 0.35E-03 | 0.89E-04 | 0.22E-04 | 0.56E-05 |
| 1/320 | — | 0.19E-02 | 0.62E-03 | 0.17E-03 | 0.45E-04 | 0.11E-04 |
| 1/640 | — | — | 0.95E-03 | 0.31E-03 | 0.87E-04 | 0.22E-04 |

Table 6.8: Rate of convergence for Example 6.4
$n = 64, 128, 256, 512, 1024$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/16 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/32 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/64 | 0.19E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/128 | 0.16E+01 | 0.19E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.19E+01 |
| 1/256 | — | 0.16E+01 | 0.19E+01 | 0.20E+01 | 0.20E+01 | 0.19E+01 |
| 1/512 | — | — | 0.16E+01 | 0.19E+01 | 0.20E+01 | 0.18E+01 |

Table 6.9: Max. Errors for Example 6.5

| $\varepsilon$ | n = 64 | n = 128 | n = 256 | n = 512 | n = 1024 | n = 2048 |
|---|---|---|---|---|---|---|
| 1/32 | 0.13E-03 | 0.32E-04 | 0.80E-05 | 0.20E-05 | 0.50E-06 | 0.12E-06 |
| 1/64 | 0.25E-03 | 0.62E-04 | 0.15E-04 | 0.38E-05 | 0.96E-06 | 0.24E-06 |
| 1/128 | 0.48E-03 | 0.12E-03 | 0.30E-04 | 0.74E-05 | 0.19E-05 | 0.46E-06 |
| 1/1024 | 0.38E-02 | 0.90E-03 | 0.22E-03 | 0.55E-04 | 0.14E-04 | 0.34E-05 |
| 1/2048 | 0.80E-02 | 0.18E-02 | 0.44E-03 | 0.11E-03 | 0.27E-04 | 0.68E-05 |
| 1/4096 | 0.17E-01 | 0.37E-02 | 0.87E-03 | 0.22E-03 | 0.54E-04 | 0.13E-04 |
| 1/10000 | 0.41E-01 | 0.98E-02 | 0.22E-02 | 0.52E-03 | 0.13E-03 | 0.32E-04 |

Table 6.10: Rate of convergence for Example 6.5
$n = 64, 128, 256, 512, 1024$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/32 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/64 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/128 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/1024 | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/2048 | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/4096 | 0.22E+01 | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.21E+01 |
| 1/10000 | 0.21E+01 | 0.22E+01 | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.21E+01 |

Table 6.11: Max. Errors for Example 6.6

| $\varepsilon$ | n = 64 | n = 128 | n = 256 | n = 512 | n = 1024 | n = 2048 |
|---|---|---|---|---|---|---|
| 1/32 | 0.51E-03 | 0.13E-03 | 0.32E-04 | 0.79E-05 | 0.20E-05 | 0.49E-06 |
| 1/64 | 0.10E-02 | 0.25E-03 | 0.63E-04 | 0.16E-04 | 0.40E-05 | 0.99E-06 |
| 1/128 | 0.20E-02 | 0.51E-03 | 0.13E-03 | 0.32E-04 | 0.79E-05 | 0.20E-05 |
| 1/1024 | 0.16E-01 | 0.40E-02 | 0.10E-02 | 0.25E-03 | 0.63E-04 | 0.16E-04 |
| 1/2048 | 0.34E-01 | 0.82E-02 | 0.20E-02 | 0.51E-03 | 0.13E-03 | 0.32E-04 |
| 1/4096 | 0.66E-01 | 0.16E-01 | 0.40E-02 | 0.10E-02 | 0.25E-03 | 0.63E-04 |
| 1/10000 | 0.13E+00 | 0.42E-01 | 0.99E-02 | 0.25E-02 | 0.62E-03 | 0.15E-03 |

Table 6.12: Rate of convergence for Example 6.6
$n = 64, 128, 256, 512, 1024$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/32 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/64 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/128 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/1024 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/2048 | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/4096 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/10000 | 0.17E+01 | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |

Table 6.13: Max. Errors for Example 6.7

| $\varepsilon$ | n = 64 | n = 128 | n = 256 | n = 512 | n = 1024 | n = 2048 |
|---|---|---|---|---|---|---|
| 1/16 | 0.58E-03 | 0.15E-03 | 0.37E-04 | 0.94E-05 | 0.23E-05 | 0.59E-06 |
| 1/32 | 0.23E-02 | 0.60E-03 | 0.15E-03 | 0.39E-04 | 0.96E-05 | 0.26E-05 |
| 1/64 | 0.74E-02 | 0.23E-02 | 0.60E-03 | 0.15E-03 | 0.39E-04 | 0.96E-05 |
| 1/128 | 0.20E-01 | 0.74E-02 | 0.22E-02 | 0.60E-03 | 0.15E-03 | 0.38E-04 |
| 1/256 | 0.48E-01 | 0.21E-01 | 0.74E-02 | 0.22E-02 | 0.59E-03 | 0.15E-03 |
| 1/512 | 0.85E-01 | 0.49E-01 | 0.21E-01 | 0.74E-02 | 0.22E-02 | 0.58E-03 |
| 1/1024 | 0.11E+00 | 0.86E-01 | 0.49E-01 | 0.20E-01 | 0.73E-02 | 0.22E-02 |

Table 6.14: Rate of convergence for Example 6.7
$n = 64, 128, 256, 512, 1024$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/2 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/4 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/8 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/16 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/32 | 0.19E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.19E+01 | 0.20E+01 |
| 1/64 | 0.17E+01 | 0.19E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.19E+01 |
| 1/128 | 0.15E+01 | 0.17E+01 | 0.19E+01 | 0.20E+01 | 0.20E+01 | 0.18E+01 |

Table 6.15: Max. Errors for Example 6.8
With Uniform Mesh

| $\varepsilon$ | $n = 32$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ |
|---|---|---|---|---|---|---|
| 1/5 | 0.13E-03 | 0.31E-04 | 0.79E-05 | 0.20E-05 | 0.58E-06 | 0.33E-06 |
| 1/10 | 0.19E-03 | 0.48E-04 | 0.12E-04 | 0.31E-05 | 0.92E-06 | 0.60E-06 |
| 1/20 | 0.33E-03 | 0.83E-04 | 0.21E-04 | 0.52E-05 | 0.14E-05 | 0.11E-05 |
| 1/40 | 0.92E-03 | 0.23E-03 | 0.58E-04 | 0.14E-04 | 0.36E-05 | 0.20E-05 |
| 1/80 | 0.31E-02 | 0.77E-03 | 0.19E-03 | 0.49E-04 | 0.12E-04 | 0.41E-05 |
| 1/100 | 0.47E-02 | 0.11E-02 | 0.29E-03 | 0.73E-04 | 0.18E-04 | 0.53E-05 |
| 1/200 | 0.00E+00 | 0.42E-02 | 0.10E-02 | 0.26E-03 | 0.65E-04 | 0.16E-04 |

Table 6.16: Max. Errors for Example 6.8
With about 12.5% mesh points in the boundary layer region

| $\varepsilon$ | $n = 32$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ |
|---|---|---|---|---|---|---|
| 1/5 | 0.52E-04 | 0.13E-04 | 0.33E-05 | 0.82E-06 | 0.21E-06 | 0.51E-07 |
| 1/10 | 0.14E-03 | 0.36E-04 | 0.90E-05 | 0.23E-05 | 0.57E-06 | 0.14E-06 |
| 1/20 | 0.39E-03 | 0.97E-04 | 0.24E-04 | 0.60E-05 | 0.15E-05 | 0.38E-06 |
| 1/40 | 0.10E-02 | 0.26E-03 | 0.64E-04 | 0.16E-04 | 0.40E-05 | 0.10E-05 |
| 1/80 | 0.28E-02 | 0.69E-03 | 0.17E-03 | 0.43E-04 | 0.11E-04 | 0.27E-05 |
| 1/100 | 0.39E-02 | 0.94E-03 | 0.23E-03 | 0.58E-04 | 0.14E-04 | 0.36E-05 |
| 1/200 | 0.13E-01 | 0.27E-02 | 0.64E-03 | 0.16E-03 | 0.40E-04 | 0.10E-04 |

Table 6.17: Max. Errors for Example 6.8
With about 25% mesh points in the boundary layer region

| $\varepsilon$ | $n = 32$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ |
|---|---|---|---|---|---|---|
| 1/5 | 0.68E-04 | 0.17E-04 | 0.43E-05 | 0.11E-05 | 0.27E-06 | 0.67E-07 |
| 1/10 | 0.18E-03 | 0.46E-04 | 0.12E-04 | 0.29E-05 | 0.72E-06 | 0.18E-06 |
| 1/20 | 0.47E-03 | 0.12E-03 | 0.29E-04 | 0.73E-05 | 0.18E-05 | 0.46E-06 |
| 1/40 | 0.11E-02 | 0.28E-03 | 0.71E-04 | 0.18E-04 | 0.45E-05 | 0.11E-05 |
| 1/80 | 0.26E-02 | 0.63E-03 | 0.16E-03 | 0.40E-04 | 0.10E-04 | 0.25E-05 |
| 1/100 | 0.32E-02 | 0.79E-03 | 0.20E-03 | 0.50E-04 | 0.13E-04 | 0.31E-05 |
| 1/200 | 0.53E-02 | 0.14E-02 | 0.36E-03 | 0.91E-04 | 0.23E-04 | 0.57E-05 |

Table 6.18: Rate of convergence for Example 6.8
With about 25% mesh points in the boundary layer region
$n = 32, 64, 128, 256, 512$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/5 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/10 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/20 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/40 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/80 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/100 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/200 | 0.19E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |

Table 6.19: Max. Errors for Example 6.9
With Uniform Mesh

| $\varepsilon$ | $n=8$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ |
|---|---|---|---|---|---|---|
| 1/32 | 0.90E-02 | 0.39E-02 | 0.11E-02 | 0.43E-03 | 0.75E-04 | 0.25E-04 |
| 1/64 | 0.29E-01 | 0.73E-02 | 0.23E-02 | 0.14E+01 | 0.14E+01 | 0.14E-01 |
| 1/128 | 0.69E-01 | 0.12E-01 | 0.32E-02 | 0.14E+01 | 0.14E+01 | 0.37E-02 |
| 1/1024 | 0.10E+00 | 0.11E-01 | 0.30E-01 | 0.16E+01 | 0.16E+01 | 0.15E-02 |
| 1/2048 | 0.85E-01 | 0.95E-01 | 0.70E-01 | 0.12E+01 | 0.12E+01 | 0.74E-03 |
| 1/4096 | 0.23E+00 | 0.42E-01 | 0.19E-01 | 0.11E+01 | 0.11E+01 | 0.20E-02 |
| 1/10000 | 0.25E+00 | 0.12E+00 | 0.11E+00 | 0.11E+01 | 0.10E+01 | 0.42E-02 |

Table 6.20: Max. Errors for Example 6.9
With about 12.5% mesh points in the boundary layer region

| $\varepsilon$ | $n=8$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ |
|---|---|---|---|---|---|---|
| 1/32 | 0.58E-01 | 0.88E-02 | 0.21E-02 | 0.24E-03 | 0.62E-04 | 0.26E-04 |
| 1/64 | 0.75E-01 | 0.15E-01 | 0.59E-02 | 0.78E-03 | 0.52E-03 | 0.35E-05 |
| 1/128 | 0.31E-01 | 0.26E-01 | 0.77E-02 | 0.13E-02 | 0.65E-03 | 0.68E-04 |
| 1/1024 | 0.75E-01 | 0.68E-01 | 0.10E-01 | 0.28E-02 | 0.71E-03 | 0.18E-03 |
| 1/2048 | 0.89E-01 | 0.72E-01 | 0.13E-01 | 0.35E-02 | 0.17E-02 | 0.21E-03 |
| 1/4096 | 0.84E-01 | 0.64E-01 | 0.15E-01 | 0.65E-02 | 0.93E-03 | 0.51E-03 |
| 1/10000 | 0.59E-01 | 0.40E-01 | 0.21E-01 | 0.90E-02 | 0.12E-02 | 0.32E-03 |

Table 6.21: Max. Errors for Example 6.9
With about 25% mesh points in the boundary layer region

| $\varepsilon$ | $n=8$ | $n=16$ | $n=32$ | $n=64$ | $n=128$ | $n=256$ |
|---|---|---|---|---|---|---|
| 1/32 | 0.10E-01 | 0.28E-02 | 0.80E-03 | 0.15E-03 | 0.64E-04 | 0.13E-06 |
| 1/64 | 0.16E-01 | 0.69E-02 | 0.93E-03 | 0.63E-03 | 0.59E-04 | 0.10E-04 |
| 1/128 | 0.28E-01 | 0.79E-02 | 0.14E-02 | 0.76E-03 | 0.25E-03 | 0.24E-04 |
| 1/1024 | 0.70E-01 | 0.11E-01 | 0.28E-02 | 0.75E-03 | 0.20E-03 | 0.10E-03 |
| 1/2048 | 0.62E-01 | 0.13E-01 | 0.37E-02 | 0.16E-02 | 0.22E-03 | 0.53E-04 |
| 1/4096 | 0.56E-01 | 0.16E-01 | 0.69E-02 | 0.83E-03 | 0.52E-03 | 0.60E-04 |
| 1/10000 | 0.40E-01 | 0.21E-01 | 0.91E-02 | 0.11E-02 | 0.33E-03 | 0.70E-04 |

Table 6.22: Rate of convergence for Example 6.9
With about 25% mesh points in the boundary layer region
$n = 8, 16, 32, 64, 128$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/32 | 0.19E+01 | 0.18E+01 | 0.24E+01 | 0.12E+01 | 0.90E+01 | 0.33E+01 |
| 1/64 | 0.12E+01 | 0.29E+01 | 0.56E+00 | 0.34E+01 | 0.26E+01 | 0.21E+01 |
| 1/128 | 0.18E+01 | 0.25E+01 | 0.89E+00 | 0.16E+01 | 0.34E+01 | 0.20E+01 |
| i/1024 | 0.27E+01 | 0.19E+01 | 0.19E+01 | 0.19E+01 | 0.99E+00 | 0.19E+01 |
| 1/2048 | 0.23E+01 | 0.18E+01 | 0.12E+01 | 0.28E+01 | 0.21E+01 | 0.20E+01 |
| 1/4096 | 0.18E+01 | 0.12E+01 | 0.30E+01 | 0.69E+00 | 0.31E+01 | 0.20E+01 |
| 1/10000 | 0.92E+00 | 0.12E+01 | 0.30E+01 | 0.18E+01 | 0.23E+01 | 0.18E+01 |

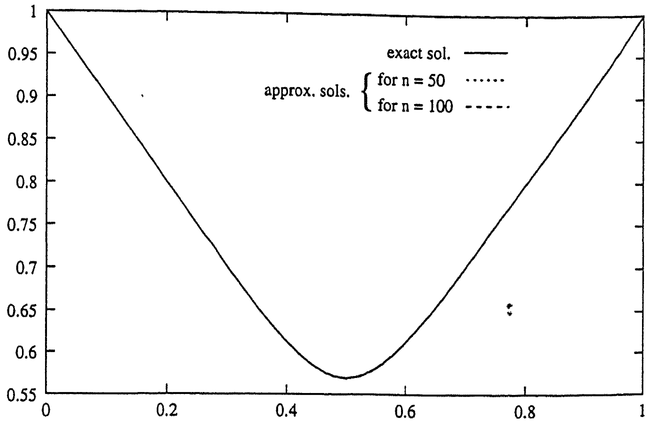


Figure 6.1: Exact and Approx. Solutions of Example 6.4 for $\varepsilon = 0.1$ and $n = 50, 100$

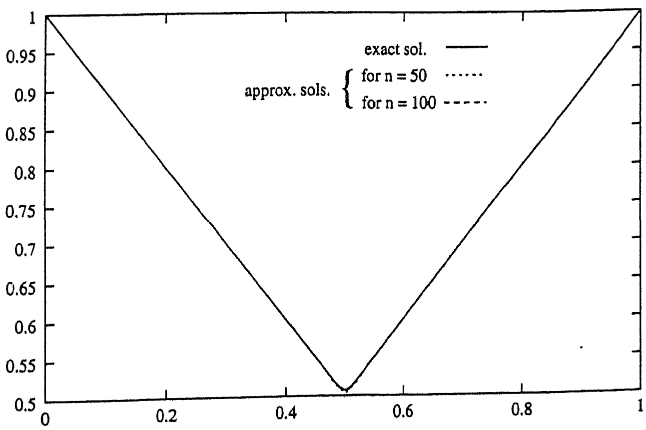


Figure 6.2: Exact and Approx. Solutions of Example 6.4 for $\varepsilon = 0.01$ and $n = 50, 100$

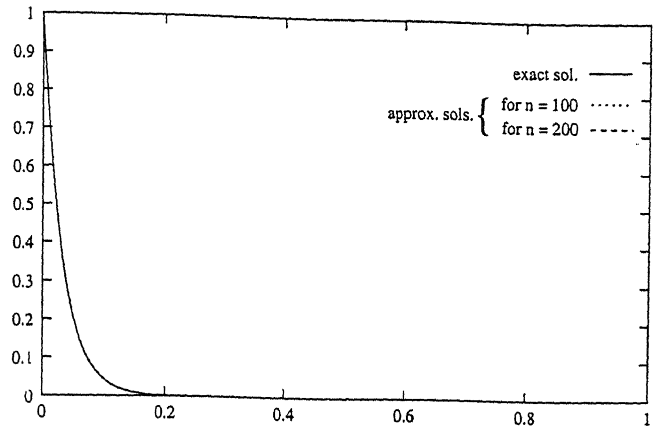


Figure 6.3: Exact and Approx. Solutions of Example 6.6 for $\varepsilon = 0.001$ and $n = 100, 200$

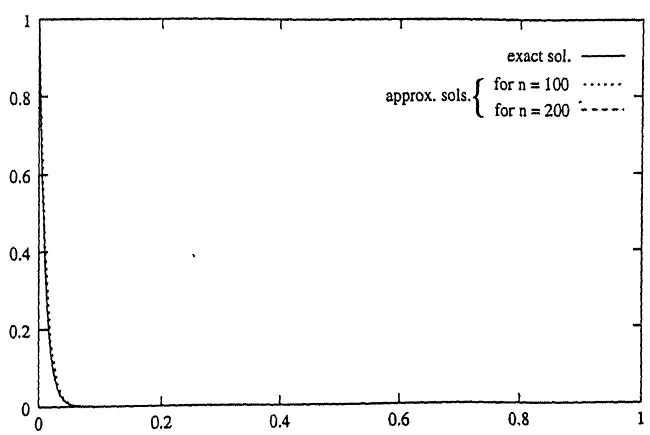


Figure 6.4: Exact and Approx. Solutions of Example 6.6 for $\varepsilon = 0.0001$ and $n = 100, 200$

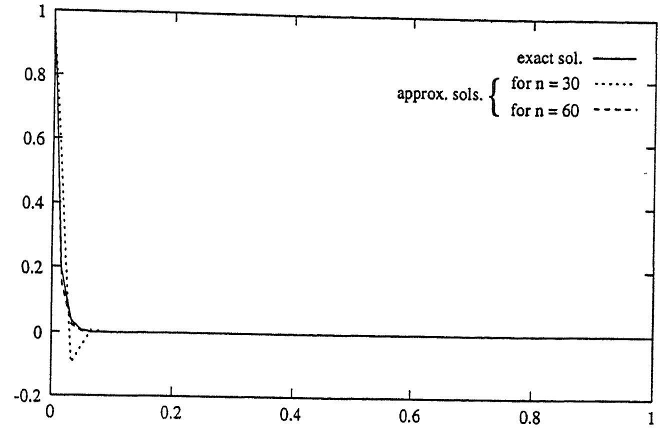


Figure 6.5: Exact and Approx. Solutions of Example 6.9 for $\varepsilon = 0.0001$ and $n = 30, 60$ With Uniform Mesh

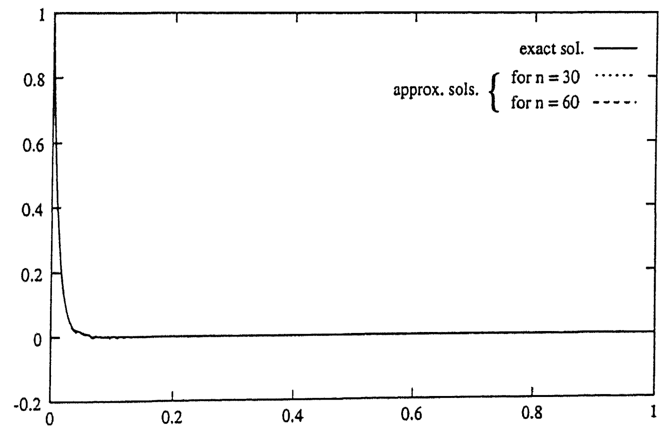


Figure 6.6: Exact and Approx. Solutions of Example 6.9 for $\varepsilon = 0.0001$ and $n = 30, 60$ With about 50 % points in B. Layer

To further corroborate the applicability of the proposed method, graphs have been plotted for few examplesfor values of $x \in [0,1]$ versus the computed (termed as approximate) solutions obtained at different values of $x$ for a fixed $\varepsilon$. Figures 6.1 - 6.2 are the graph for Example 6.4 for $\varepsilon = 0.1$ and 0.01, respectively. For each of these plots we took $n = 50$ and 100. Figures 6.3 - 6.4 are the graph for Example 6.6 for $\varepsilon = 0.001$ and 0.0001, respectively. For each plot we took $n = 100$ and 200. Figure 6.5 is the graph for Example 6.9 with uniform mesh whereas Figure 6.6 is plotted by taking about 50 % mesh points in the boundary layer region.

It can be seen from the Figures 6.1 - 6.4 that the approximate solutions are perfactly identicle to the exact solution for each of these two cases. Also from the Figure 6.5 it can be observed that the exact and approximate solutions with uniform mesh are identical for most of the range except in the boundary layer region where these two solutions deviate from each other. To control these fluctuations, we took more mesh points in the boundary layer and the resulting behaviour can be seen from the Figure 6.6 for the corresponding value of $\varepsilon$.

The consistency condition (used in the method SC) needs the parameter(s) to be small. Therefore in the Tables 6.1 - 6.8, we included the results corresponding to only those values of $h$ and $\varepsilon$ which satisfy this requirement.

Our mesh selection procedure (used in the method VMCS) needs prior knowledge of $\delta$, $\widetilde{h}_1$ and $K$. For Example 6.8 we have chosen $\delta = 0.05$ whereas for Example 6.9 we took $\delta = O(\sqrt{\varepsilon})$. The other two parameters have been taken as $\widetilde{h}_1 = 0.0001$, $K = 0.1$ for Example 6.8 and $\widetilde{h}_1 = 0.001$, $K = 0.1$ for Example 6.9. However the increase in $K$ will lead to more concentration of points near the boundaries. Moreover, for a fixed $K$, the increase in $\widetilde{h}_1$ leads to the same conclusion.

# Chapter 7

## VARIABLE MESH SPLINE APPROXIMATION METHOD FOR SINGULARLY PERTURBED TURNING POINT PROBLEMS

## 7.1 Introduction

We consider singularly perturbed two point boundary value problems which are mathematical models of diffusion-convection processes or related physical phenomena. The diffusion term is the term involving the second order derivative and the convective term is that involving the first order derivative. In many practical problems the coefficient of the second derivative term is small compared to the coefficient of the first derivative term. Examples of these are heat transport problems with large Peclet numbers, Navier-Stokes flows with large Reynolds numbers and magneto-hydrodynamic duct problems at high Hartman numbers. These problems are at the parabolic-hyperbolic or parabolic-elliptic interface and thus exhibit some of the features of the differential equations. Depending on the nature of the coefficients of the first and second derivative terms we may have either boundary or interior layers for the related turning point problem (TPP).

In this chapter we consider two type of problems:

(i) Turning point problems having boundary layers, and

(ii) Turning point problems having interior layers.

Consider the following class of singularly perturbed turning point problem

$$\left.\begin{array}{c} Ly \equiv \varepsilon y'' + a(x)y' - b(x)y = f(x) \quad \text{on} \quad [p_1, p_2] \\ y(p_1) = \eta_1, \quad y(p_2) = \eta_2 \end{array}\right\} \tag{7.1}$$

where, $a(x)$ is assumed to be in $C^2[p_1, p_2]$, $b(x)$ and $f(x)$ are required to be in $C^1[p_1, p_2]$, $\eta_1$, $\eta_2$ are given constants, $p_1 \le 0$, $p_2 > 0$ (usually $p_1 = -1$ and $p_2 = 1$), $0 < \varepsilon << 1$.

Moreover

$$a(0) = 0 \quad , \quad a'(0) < 0 \tag{7.2}$$

or

$$a(0) = 0 \quad , \quad a'(0) > 0 \tag{7.3}$$

In order that the solution of (7.1) satisfies a maximum principle, we require that

$$b(x) \geq 0 \quad , \quad b(0) > 0 \tag{7.4}$$

Also $b(x)$ is required to be bounded below by some positive constant $b$, i. e.,

$$b(x) \geq b > 0 \tag{7.5}$$

to exclude the so-called resonance cases [2]. We also impose the following restriction which ensures that there are no other turning points in the interval $[p_1, p_2]$:-

$$|a'(x)| \geq \left|\frac{a'(0)}{2}\right| \quad , \quad x \in [p_1, p_2] \tag{7.6}$$

(If there is no first derivative term but if $b(x)$ changes sign then also turning point occurs, conventionally termed as classical turning point).

Under the conditions (7.2), (7.4) - (7.6), the turning point problem (7.1) has a unique solution having two boundary layers at $x = p_1$ and $x = p_2$ whereas under the conditions (7.3), (7.4) - (7.6), the turning point problem (7.1) has a unique solution having interior layer.

The turning point is simple if $a(x)$ vanishes at $x = 0$ and is called a multiple turning point if not only $a(x)$ but its first derivative as well vanishes at $x = 0$. Simple turning point problems have attracted the most attention of all turning point problems, both analytically and numerically. The present work deals with the simple turning point problems. However, for multiple turning point problems, one can see, e.g., [27], [262], etc.

There are three principle approaches to solve such problems numerically, namely, the Finite-Difference Methods, the Finite-Element Methods and the Spline Approximation Methods. A rigorous analysis based on comparison function approach (using finite difference techniques) can be found in Berger et al. [27]. Kellogg [132] solved these type of

problems using Allen-Southwell difference schemes. Farrell [78] derived sufficient conditions for uniform convergence. Sun and Stynes [234] used Galerkin finite element methods for such problems, whereas Surla and Uzelac [241] solved them by taking a linear combination of the two spline difference schemes: El Mistikawy and Werle (EMW) scheme of [27] and Improved El Mistikawy and Werle (IEMW) scheme of [240]. Other results for numerical methods for turning point problems have been obtained in [139], [169] and [189].

In this paper we have used the third approach, namely, the Spline Approximation Method, to solve the problems of the type (7.1). To reduce the volume of computations, we introduce a non-uniform mesh in the boundary/interior layer region(s) and uniform outside these regions.

There are two possibilities to obtain small truncation error inside the boundary/interior layer(s). The first is to choose a fine mesh there whereas the second one is to choose a difference formula reflecting the behaviour of the solution(s) inside the boundary/interior layer(s). Since the spline difference scheme has the same order of precision and the same matrix structure on the uniform and on the non-uniform grid for a fixed $\varepsilon$, we used this property for singularly perturbed problems. This enables us in modifying of the distribution of mesh points vis-a-vis to the properties of the exact solution.

In Section 7.2 we give a brief description of the method. The derivation of the difference schemes has been given in Section 7.3. Meshes are chosen according to the mesh selection strategy given in Section 7.4. In Section 7.5, the second order accuracy of the method, in the case of twin boundary layers and for interior layer problems is shown. To demonstrate the applicability of the proposed method several numerical examples have been solved in Section 7.6 and the results are presented along with their comparison with the results obtained by using uniform mesh. Our numerical results show that the same order of convergence, as in the case of twin boundary layers, is obtained in the cases of only one boundary layer (either to the left end or to the right end of the underlying interval) also. Finally, the discussion on these numerical results is presented in Section 7.7.

## 7.2 Description of the Method

The approximate solution of the problem (7.1) is sought in the form of the cubic spline function, which on each interval $[x_{j-1}, x_j]$, denoted by $S_j(x)$ and will be defined as follows:

Let

$$x_0 = p_1 \ , \ x_j = p_1 + \sum_{m=1}^{j} h_m \ , j = 1(1)n, \ h_m = x_m - x_{m-1}, \ x_n = p_2$$

For the values $y(x_0)$, $y(x_1)$, $\cdots$ , $y(x_n)$, there exists an interpolating cubic spline with the following properties:

(i) $S_j(x)$ coincides with a polynomial of degree 3 on each interval $[x_{j-1}, x_j], j = 1(1)n$

(ii) $S_j(x) \in C^2[p_1, p_2]$

(iii) $S_j(x_j) = y(x_j), \ j = 0(1)n$

Hence as in [7], the cubic spline can be given as:

$$\begin{aligned} S_j(x) \ &= \ \frac{(x_j - x)^3}{6h_j} M_{j-1} + \frac{(x - x_{j-1})^3}{6h_j} M_j + \left( y_{j-1} - \frac{h_j^2 M_{j-1}}{6} \right) \left( \frac{x_j - x}{h_j} \right) \\ & \quad + \left( y_j - \frac{h_j^2 M_j}{6} \right) \left( \frac{x - x_{j-1}}{h_j} \right) \end{aligned} \tag{7.7}$$

where,

$$x \in [x_{j-1}, x_j] \ , \ h_j = x_j - x_{j-1} \ , \ j = 1(1)n$$

and

$$M_j = S_j''(x_j) \ , \ j = 0(1)n$$

Using this spline function we will derive the difference scheme in Section 7.3, which will give us the approximate solution of $y(x)$.

## 7.3 Derivation of the Scheme

Differentiating (7.7) and denoting the approximate solution to $y(x)$ by $u(x)$, we get

$$S_j'(x) = -\frac{(x_j - x)^2}{2h_j} M_{j-1} + \frac{(x - x_{j-1})^2}{2h_j} M_j + \left( \frac{u_j - u_{j-1}}{h_j} \right) - \left( \frac{M_j - M_{j-1}}{6} \right) h_j \tag{7.8}$$

Since $S_j(x) \in C^2[0,1]$, therefore we have

$$S_j'(x_j) = S_{j+1}'(x_j) \tag{7.9}$$

$$\Rightarrow \quad \frac{h_j}{6}M_{j-1} + \frac{h_j + h_{j+1}}{6}M_j + \frac{h_{j+1}}{6}M_{j+1} = \frac{u_{j+1} - u_j}{h_{j+1}} - \frac{u_j - u_{j-1}}{h_j} \tag{7.10}$$

where

$$M_j = \frac{1}{\varepsilon}\left(f_j - a_j u_j' + b_j u_j\right) \tag{7.11}$$

Taking the Taylor series expansion for $u$ around $x_j$ and neglecting the terms containing third and higher order terms, we get the following approximations for $u_{j+1}$ and $u_{j-1}$ :

$$u_{j+1} \approx u_j + h_{j+1}u_j' + \frac{h_{j+1}^2}{2}u_j'' \tag{7.12}$$

$$u_{j-1} \approx u_j - h_j u_j' + \frac{h_j^2}{2}u_j'' \tag{7.13}$$

From equations (7.12) and (7.13), we get

$$u_j' = \frac{1}{h_j h_{j+1}\left(h_j + h_{j+1}\right)}\left[h_j^2 u_{j+1} - \left(h_j^2 - h_{j+1}^2\right)u_j - h_{j+1}^2 u_{j-1}\right] \tag{7.14}$$

and

$$u_j'' = \frac{2}{h_j h_{j+1}\left(h_j + h_{j+1}\right)}\left[h_j u_{j+1} - \left(h_j + h_{j+1}\right)u_j - h_{j+1}u_{j-1}\right] \tag{7.15}$$

Also

$$u_{j+1}' \approx u_j' + h_{j+1}u_j'' \tag{7.16}$$

and

$$u_{j-1}' \approx u_j' - h_j u_j'' \tag{7.17}$$

From equations (7.14), (7.15) and (7.16), we get

$$u_{j+1}' \approx \frac{1}{h_j h_{j+1}\left(h_j + h_{j+1}\right)}\left[h_{j+1}^2 u_{j-1} - \left(h_j + h_{j+1}\right)^2 u_j + \left(h_j^2 + 2h_j h_{j+1}\right)u_{j+1}\right] \tag{7.18}$$

and from equations (7.14), (7.15) and (7.17), we get

$$u_{j-1}' \approx \frac{1}{h_j h_{j+1}\left(h_j + h_{j+1}\right)}\left[-\left(h_{j+1}^2 + 2h_j h_{j+1}\right)u_{j-1} - \left(h_j + h_{j+1}\right)^2 u_j - h_j^2 u_{j+1}\right] \tag{7.19}$$

Therefore, using (7.10), (7.11), (7.14), (7.18) and (7.19), we obtain the difference scheme

$$Ru_j = QZ_j \ , \ j = 1(1)n-1 \tag{7.20}$$

where,

$$Ru_j = r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1}$$

$$QZ_j = q_j^- Z_{j-1} + q_j^c Z_j + q_j^+ Z_{j+1}$$

$$u_0 = \alpha_0 \ , \ u_n = \alpha_1$$

$$r_j^- = \frac{2h_j + h_{j+1}}{6(h_j + h_{j+1})} a_{j-1} + \frac{h_{j+1}}{3h_j} a_j - \frac{h_{j+1}^2}{6h_j(h_j + h_{j+1})} a_{j+1} + \frac{h_j}{6} b_{j-1} - \frac{\varepsilon}{h_j}$$

$$r_j^+ = \frac{h_j^2}{6h_{j+1}(h_j + h_{j+1})} a_{j-1} - \frac{h_j}{3h_{j+1}} a_j - \frac{2h_{j+1} + h_j}{6(h_j + h_{j+1})} a_{j+1} + \frac{h_{j+1}}{6} b_{j+1} - \frac{\varepsilon}{h_{j+1}}$$

$$r_j^c = -\frac{h_j + h_{j+1}}{6h_{j+1}} a_{j-1} - \frac{h_{j+1}^2 - h_j^2}{3h_j h_{j+1}} a_j + \frac{h_j + h_{j+1}}{6h_j} a_{j+1} + \frac{h_j + h_{j+1}}{3} b_j + \frac{\varepsilon}{h_j} + \frac{\varepsilon}{h_{j+1}}$$

$$q_j^- = -\frac{h_j}{6}, \ q_j^+ = -\frac{h_{j+1}}{6}, \ q_j^c = -\frac{h_j + h_{j+1}}{3} \tag{7.21}$$

# 7.4 Mesh Selection Strategies

## 7.4.1 Mesh Selection Strategy for TPPs Having Boundary Layer(s)

We form the non-uniform grid in such a way that more points are generated in the boundary layer regions than outside these regions.

Let the concerned interval on which the problem is to be solved is $[p_1, p_2]$, where $p_1 < 0$ and $p_2 > 0$. Let $\delta$ denotes the width of the boundary layer.

We have three cases :-

Case I : There are two boundary layers (one at both ends).

Case II : There is one boundary layer at the left end.

Case III : There is one boundary layer at the right end.

First we consider the case of two boundary layers.

In this case we have three subintervals:

$$[p_1 \, , \, p_1 + \delta] \quad , \quad [p_1 + \delta \, , \, p_2 - \delta] \ \text{ and } \ [p_2 - \delta \, , \, p_2]$$

Let $n_1$, $n_2$ and $n_3$ be the number of points in these subintervals respectively such that $n_1 + n_2 + n_3 = n$ and $n_1 = n_3$. Further let the positive constants $\widetilde{h}_1$ and $K$ be known. Then we generate the mesh as follows:

On the interval $[p_1\,,\, p_1 + \delta]$, the grid is non-uniform and is defined as follows:

$\widetilde{h}_j = \widetilde{h}_{j-1} + K\left[\frac{\widetilde{h}_{j-1}}{\varepsilon}\right] \min\left(\widetilde{h}^2_{j-1}, \varepsilon\right) \quad , \quad j = 2(1)n_1.$

Now, let

$\widetilde{q} = \sum_{j=1}^{n_1} \widetilde{h}_j$

$q = \frac{\delta}{\widetilde{q}}$

$h_j = q\widetilde{h}_j \quad , \quad j = 1(1)n_1$

On the interval $[p_1 + \delta\,,\, p_2 - \delta]$, the grid is uniform and is defined as follows:

$h_j = \frac{(p_2 - p_1 - 2\delta)}{n_2} \quad , \quad j = n_1 + 1\,, n_1 + n_2.$

On the interval $[p_2 - \delta\,,\, p_2]$ the grid is the mirror image of the grid on $[p_1\,,\, p_1 + \delta]$ and therefore will be given by:

$h_j = h_{n+1-j} \quad , \quad j = n_1 + n_2 + 1\,,\, n.$

and define

$x_0 = p_1$

$x_j = x_{j-1} + h_j \quad , \quad j = 1(1)n.$

Now we consider the second case in which there is one boundary layer at the left end of the underlying interval.

In this case we have two subintervals, viz.,

$$[p_1\,,\, p_1 + \delta] \quad \text{and} \quad [p_1 + \delta\,,\, p_2]$$

Let $n_1$ and $n_2$ be the number of points in these two subintervals respectively such that $n_1 + n_2 = n$. We define non-uniform mesh in the interval $[p_1\,,\, p_1 + \delta]$ in a similar manner as we did in the first case and then in the subinterval $[p_1 + \delta\,,\, p_2]$ we define uniform mesh which will be given by:

$$h_j = \frac{(p_2 - p_1 - \delta)}{n_2} \quad , \quad j = n_1 + 1\,,\, n$$

Finally, we consider the third case in which there is one boundary layer at the right end of the underlying interval. Unlike the second case, in this case we consider the following three subintervals :

$$[p_1\,,\, p_1 + \delta] \quad , \quad [p_1 + \delta\,,\, p_2 - \delta] \quad \text{and} \quad [p_2 - \delta\,,\, p_2]$$

As in the first case here also we define non-uniform mesh in the interval $[p_1\,,\,p_1+\delta]$. Then we take its mirror image in the interval $[p_2-\delta\,,\,p_2]$ ans since we need fine mesh near right end only, therefore we define uniform grid in the interval $[p_1\,,\,p_2-\delta]$.

### 7.4.2 Mesh Selection Strategy for TPPs Having Interior Layers

We form the non-uniform grid in such a way that more points are generated in the interior layer region than outside this.

Let the concerned interval on which the problem is to be solved is $[p_1, p_2]$, where $p_1 < 0$ and $p_2 > 0$. Let its midpoint, i.e., $(p_1+p_2)/2$, be denoted by $c_p$. Let $\delta$ denotes the width of the interior layer. Thus we have four subintervals:

$$\left[p_1\,,\,c_p-\frac{\delta}{2}\right]\quad,\quad\left[c_p-\frac{\delta}{2}\,,\,c_p\right]\quad,\quad\left[c_p\,,\,c_p+\frac{\delta}{2}\right]\ \text{ and }\ \left[c_p+\frac{\delta}{2}\,,\,p_2\right]$$

Let $n_1$, $n_2$, $n_3$ and $n_4$ be the number of points in these subintervals respectively $n_1+n_2+n_3+n_4=n$, $n_1=n_4$ and $n_2=n_3$.Further let the positive constants $\widetilde{h}_{n_1+n_2+1}$ and $K$ be known. Then we generate the mesh as follows:

On the interval $\left[p_1\,,\,c_p-\frac{\delta}{2}\right]$, the grid is uniform and is defined as follows:

$h_j = \frac{\left(c_p-\frac{\delta}{2}\right)-p_1}{n_1} \quad , \quad j=1(1)n_1.$

On the interval $\left[c_p\,,\,c_p+\frac{\delta}{2}\right]$, the grid is non-uniform and is defined as follows:

$\widetilde{h}_{n_1+n_2+j} = \widetilde{h}_{n_1+n_2+j-1} + K\left[\frac{\widetilde{h}_{n_1+n_2+j-1}}{\varepsilon}\right]\min\left(\widetilde{h}^2_{n_1+n_2+j-1}, \varepsilon\right) \quad , \quad j=2(1)n_3.$

Now, let

$\widetilde{q} = \sum_{j=1}^{n_3} \widetilde{h}_{n_1+n_2+j}$

$q = \frac{\delta/2}{\widetilde{q}}$

$h_{n_1+n_2+j} = q\widetilde{h}_{n_1+n_2+j} \quad , \quad j=1(1)n_3$

On the interval $\left[c_p-\frac{\delta}{2}\,,\,c_p\right]$ the grid is the mirror image of the grid on $\left[c_p\,,\,c_p+\frac{\delta}{2}\right]$ and therefore will be given by:

$h_{n_1+n_2-j+1} = h_{n_1+n_2+j} \quad , \quad j=1(1)n_3.$

Finally, on the interval $\left[c_p+\frac{\delta}{2}\,,\,p_2\right]$, the grid is uniform and is defined as follows:

$h_j = \frac{p_2-\left(c_p-\frac{\delta}{2}\right)}{n_4} \quad , \quad j=n_1+n_2+n_3+1\,,\,n.$

and define

$x_0 = p_1$

$x_j = x_{j-1}+h_j \quad , \quad j=1(1)n.$

# 7.5 Proofs of the Uniform Convergence

## 7.5.1 Proof of the Uniform Convergence for TPPs Having Boundary Layer(s)

For the error analysis of the problems having twin boundary layers, we have used the comparison functions method developed by Kellogg and Tsan [131] and Berger et al. [26]. Analysis for the case of only one boundary layer (either to the left or to the right end) can be done analogously (see, e.g., Berger et al. [26], for details).

For the sake of simplicity, we consider $p_1 = -1$ and $p_2 = 1$. Also throughout the chapter $M$ will denote positive constant which may take different values in different equations(inequalities) but that are always independent of $h$ and $\varepsilon$.

This method uses the following two Lemmas [26]:

**Lemma 7.1** *(maximum principle): Let* $\{u_j\}$ *be a set of values at the grid points* $x_j$*, satisfying* $u_0 \leq 0, u_n \leq 0$ ; $(u_0 = u(x = -1))$ *and* $Ru_j \geq 0, j = 1(1)n-1$*, then* $u_j \leq 0, j = 0(1)n$.

**Lemma 7.2** *If* $K_1(h,\varepsilon) \geq 0$ *and* $K_2(h,\varepsilon) \geq 0$ *are such that*

$$R(K_1(h,\varepsilon)\phi_j + K_2(h,\varepsilon)\psi_j) \;\geq\; R(\pm e_j) \;=\; \pm\tau_j(y)$$

*for each* $j = 1, 2, \cdots, n-1$*, then the discrete maximum principle implies that*

$$|e_j| \;\leq\; K_1(h,\varepsilon)|\phi_j| \;+\; K_2(h,\varepsilon)|\psi_j|$$

*where,* $|e_j| \;=\; |y(x_j) - u_j|$*, for each* $j$ *and* $\phi$ *and* $\psi$ *are two comparison functions.*

We use the following Lemma (which is a slite extension of the Lemma 2.4 in [131]), for the properties of the exact solution of (7.1):

**Lemma 7.3** *If* $y$ *satisfies (7.1), then*

$$y(x) = v(x) + w(x) + g(x)$$

*where,*

$$v(x) = \left(-\frac{\varepsilon y'(-1)}{a(-1)}\right) exp\left(-\frac{a(-1)}{\varepsilon}(1+x)\right)$$

$$w(x) = \left(-\frac{\varepsilon y'(1)}{a(1)}\right) exp\left(-\frac{a(1)}{\varepsilon}(1-x)\right)$$

*and*

$$\left|g^{(k)}(x)\right| \leq M\left\{1+\varepsilon^{-k+1}exp\left(-\frac{2c}{\varepsilon}(1-x)\right)\right\}$$

$k = 0(1)4$, *c is some constant and M is a positive constant independent of h and* $\varepsilon$.

We use the following two comparison functions (as in [27]):

$$\phi = C_1 exp\left[-2\eta C_3\frac{x+1}{\varepsilon}\right] \quad \text{and} \quad \psi = C_2 exp\left[-2\eta C_3\frac{1-x}{\varepsilon}\right]$$

where, $C_1$, $C_2$ and $\eta$ are constants independent of $h$ and $\varepsilon$.

Remark :- The following inequalities hold :

$$R\phi_j \geq M \quad , \quad h_c^2 \leq \varepsilon$$

$$R\phi_j \geq Mh_c \quad , \quad h_c^2 \geq \varepsilon$$

$$R\psi_j \geq M \quad , \quad h_c^2 \leq \varepsilon$$

$$R\psi_j \geq Mh_c \quad , \quad h_c^2 \geq \varepsilon$$

where

$$h_c = \max_j h_j \; (= \text{a constant})$$

Now we estimate the truncation error of the scheme (7.20) using (7.21).

First consider the case in which $h_c^2 \leq \varepsilon$.

We have,

$$\tau_j(y) \;=\; T_0 y_j + T_1 y_j' + T_2 y_j'' + T_3 y_j''' + \text{Remainder terms}$$

where,

$$\begin{aligned}
T_0 &= \left(r_j^- + r_j^c + r_j^+\right) + \left(q_j^- b_{j-1} + q_j^c b_j + q_j^+ b_{j+1}\right)\\
T_1 &= \left(h_{j+1}r_j^+ - h_j r_j^-\right) - \left\{q_j^-\left(a_{j-1} + h_j b_{j-1}\right) + q_j^c a_j + q_j^+\left(a_{j+1} - h_{j+1}b_{j+1}\right)\right\}\\
T_2 &= \left(\tfrac{h_j^2}{2}r_j^- + \tfrac{h_{j+1}^2}{2}r_j^+\right) - \varepsilon\left(q_j^- + q_j^c + q_j^+\right)\\
&\quad + \left\{q_j^-\left(h_j a_{j-1} + \tfrac{h_j^2}{2}b_{j-1}\right) + q_j^+\left(-h_{j+1}a_{j+1} + \tfrac{h_{j+1}^2}{2}b_{j+1}\right)\right\}\\
T_3 &= \left(\tfrac{h_{j+1}^3}{6}r_j^+ - \tfrac{h_j^3}{6}r_j^-\right) + \varepsilon\left(q_j^- h_j - q_j^+ h_{j+1}\right) - q_j^-\left\{\tfrac{h_j^2}{2}a_{j-1} + \tfrac{h_j^3}{6}b_{j-1}\right\}\\
&\quad - q_j^+\left\{\tfrac{h_{j+1}^2}{2}a_{j+1} - \tfrac{h_{j+1}^3}{6}b_{j+1}\right\}
\end{aligned}$$

Using (7.21) we see that $T_0 = 0, T_1 = 0, T_2 = 0$ and $|T_3| \leq Mh_c^3$

Now from Lemma 7.3, we have

$$v_j''' = \left(-\frac{a(-1)}{\varepsilon}\right)^2 y'(-1)\, exp\left\{-\frac{a(-1)(1+x_j)}{\varepsilon}\right\}$$

therefore

$$|\tau_j(v)| \leq \frac{Mh_c^3}{\varepsilon^2} exp\left\{-\frac{a(-1)(1+x_j)}{\varepsilon}\right\} \ , \ h_c^2 \leq \varepsilon$$

Similarly,

$$|\tau_j(w)| \leq M\frac{h_c^3}{\varepsilon^2} exp\left\{-\frac{a(1)(1-x_j)}{\varepsilon}\right\} \ , \ h_c^2 \leq \varepsilon$$

Also

$$\left|g_j^{(3)}\right| \leq M\left[1+\varepsilon^{-2} exp\left\{-\frac{2c(1-x_j)}{\varepsilon}\right\}\right]$$

$$\Rightarrow |\tau_j(g)| \leq Mh_c^3\left[1+\frac{1}{\varepsilon^2} exp\left\{-\frac{2c(1-x_j)}{\varepsilon}\right\}\right] \ , \ h_c^2 \leq \varepsilon$$

Now

$$\tau_j(y) = \tau_j(v)+\tau_j(w)+\tau_j(g)$$

$$\Rightarrow \quad |\tau_j(y)| \leq M\frac{h_c^3}{\varepsilon^2}\left[1+exp\left\{-\frac{a(-1)(1+x_j)}{\varepsilon}\right\}+exp\left\{-\frac{a(1)(1-x_j)}{\varepsilon}\right\}\right] \ , \ h_c^2 \leq \varepsilon$$

In the opposite case, i.e., when $h_c^2 \geq \varepsilon$, we use the following expression for truncation error:

$$\begin{aligned}\tau_j(y) = &\left(\frac{h_{j+1}^3}{6}r_j^+ - \frac{h_j^3}{6}r_j^-\right)y'''(\xi_1) + \varepsilon\left(q_j^- h_j - q_j^+ h_{j+1}\right)y'''(\xi_2)\\ &-q_j^-\left\{\frac{h_j^2}{2}a_{j-1}+\frac{h_j^3}{6}b_{j-1}\right\}y'''(\xi_3) - q_j^+\left\{\frac{h_{j+1}^2}{2}a_{j+1}-\frac{h_{j+1}^3}{6}b_{j+1}\right\}y'''(\xi_4)\end{aligned}$$

$$x_{j-1} < \xi_i < x_{j+1} \ , \ i = 1(1)4$$

Now

$$\left|\frac{h_{j+1}^3}{6}r_j^+ - \frac{h_j^3}{6}r_j^-\right| \leq Mh_c^3 \ , \ \left|\varepsilon\left(q_j^- h_j - q_j^+ h_{j+1}\right)\right| \leq Mh_c^3$$

$$\left|q_j^-\left(\frac{h_j^2}{2}a_{j-1}+\frac{h_j^3}{6}b_{j-1}\right)\right| \leq Mh_c^3 \ , \ \left|q_j^+\left(\frac{h_{j+1}^2}{2}a_{j+1}-\frac{h_{j+1}^3}{6}b_{j+1}\right)\right| \leq Mh_c^3$$

Using these estimates and the above expression for $\tau_j(y)$, we obtain the same estimates for $\tau_j(v)$, $\tau_j(w)$ and $\tau_j(g)$ as were in the case of $h_c^2 \leq \varepsilon$. Choosing

$$K_1 = h_c^2 exp\left\{-\frac{a(-1)(1+x_j)}{\varepsilon}\right\} \quad \text{and} \quad K_2 = h_c^2 exp\left\{-\frac{a(1)(1-x_j)}{\varepsilon}\right\}$$

we see that the Lemma 7.2 is satisfied(in both the cases $h_c^2 \leq \varepsilon$ and $h_c^2 \geq \varepsilon$) and thus we have proved the following theorem:

**Theorem 7.1** *Let $\{u_j\}$, $j = 0(1)n$, be a set of values of the approximate solution to $y(x)$ of (7.1), obtained using (7.20) and (7.21). Then there are positive constants $\widetilde{C}_1$, $\widetilde{C}_2$ and $M$ (independent of $h$ and $\varepsilon$) such that the following estimate holds:*

$$\max_j |y_j - u_j| \leq M h_c^2 \left[ exp\left\{ -\frac{\widetilde{C}_1(1+x_j)}{\varepsilon} \right\} + exp\left\{ -\frac{\widetilde{C}_2(1-x_j)}{\varepsilon} \right\} \right]$$

where

$$h_c = \max_j h_j = \text{a constant.}$$

### 7.5.2 Proof of the Uniform Convergence for TPPs Having Interior Layer

For the crror analysis for TPPs having interior layers, we have used the same comparison functions method as we used for TPPs having boundary layers.

This method uses the same two Lemmas, viz., Lemmas 7.1 and 7.2 as we used for TPPs having boundary layers.

Additionally, we use the following Lemma (which is anologous to the Lemma 2.4 in [131]), for the properties of the exact solution of (7.1):

**Lemma 7.4** *If $y$ satisfies (7.1), then*

$$y(x) = g_1(x) + v(x) + g_2(x)$$

*where,*

$$v(x) = \left( -\frac{\varepsilon y'(\theta)}{a(\theta)} \right) exp\left[ -\frac{a(\theta)}{\varepsilon}\left(1+\frac{\delta}{2}\right) x \right]$$

$$\left| g_1^{(k)}(x) \right| \leq M \left[ 1 + \varepsilon^{-k+1} exp\left\{ -\frac{c\left(1+\frac{\delta}{2}\right)}{\varepsilon} x \right\} \right]$$

*and*

$$\left| g_2^{(k)}(x) \right| \leq M \left[ 1 + \varepsilon^{-k+1} exp\left\{ -\frac{c\left(1-\frac{\delta}{2}\right)}{\varepsilon} x \right\} \right]$$

$\theta = c_p - \frac{\delta}{2}$, $k = 0(1)4$, *$c$ is some constant and $M$ is a positive constant independent of $h$ and $\varepsilon$.*

We use the two comparison functions:

$$\phi = C_1 exp\left[-2\eta C_3 \frac{\left(1+\frac{\delta}{2}\right)x}{\varepsilon}\right] \quad \text{and} \quad \psi = C_2 exp\left[-2\eta C_3 \frac{\left(1-\frac{\delta}{2}\right)x}{\varepsilon}\right]$$

where, $C_1$, $C_2$ and $\eta$ are constants independent of $h$ and $\varepsilon$.

Remark :- The following inequalities hold :

$$R\phi_j \geq M \quad , \quad h_c^2 \leq \varepsilon$$

$$R\phi_j \geq M h_c \ , \quad h_c^2 \geq \varepsilon$$

$$R\psi_j \geq M \quad , \quad h_c^2 \leq \varepsilon$$

$$R\psi_j \geq M h_c \ , \quad h_c^2 \geq \varepsilon$$

where

$$h_c = \max_j h_j \ ( = \text{a constant})$$

Now we estimate the truncation error of the scheme (7.20) using (7.21).

First consider the case in which $h_c^2 \leq \varepsilon$.

We have,

$$\tau_j(y) \ = \ T_0 y_j + T_1 y_j' + T_2 y_j'' + T_3 y_j''' + \text{Remainder terms}$$

where,

$$\begin{aligned}
T_0 &= \left(r_j^- + r_j^c + r_j^+\right) + \left(q_j^- b_{j-1} + q_j^c b_j + q_j^+ b_{j+1}\right) \\
T_1 &= \left(h_{j+1} r_j^+ - h_j r_j^-\right) - \left\{q_j^- (a_{j-1} + h_j b_{j-1}) + q_j^c a_j + q_j^+ (a_{j+1} - h_{j+1} b_{j+1})\right\} \\
T_2 &= \left(\tfrac{h_j^2}{2} r_j^- + \tfrac{h_{j+1}^2}{2} r_j^+\right) - \varepsilon \left(q_j^- + q_j^c + q_j^+\right) \\
&\quad + \left\{q_j^- \left(h_j a_{j-1} + \tfrac{h_j^2}{2} b_{j-1}\right) + q_j^+ \left(-h_{j+1} a_{j+1} + \tfrac{h_{j+1}^2}{2} b_{j+1}\right)\right\} \\
T_3 &= \left(\tfrac{h_{j+1}^3}{6} r_j^+ - \tfrac{h_j^3}{6} r_j^-\right) + \varepsilon \left(q_j^- h_j - q_j^+ h_{j+1}\right) - q_j^- \left\{\tfrac{h_j^2}{2} a_{j-1} + \tfrac{h_j^3}{6} b_{j-1}\right\} \\
&\quad - q_j^+ \left\{\tfrac{h_{j+1}^2}{2} a_{j+1} - \tfrac{h_{j+1}^3}{6} b_{j+1}\right\}
\end{aligned}$$

Using (7.21) we see that $T_0 = 0, T_1 = 0, T_2 = 0$ and $|T_3| \le M h_c^3$

Now from Lemma 7.4, we have

$$v_j''' = \left(-\frac{a(\theta)}{\varepsilon}\right)^2 \left(1+\frac{\delta}{2}\right)^3 y'(\theta)\, exp\left\{-\frac{a(\theta)}{\varepsilon}\left(1+\frac{\delta}{2}\right)x_j\right\}$$

therefore

$$|\tau_j(v)| \;\le\; \frac{M h_c^3}{\varepsilon^2}\, exp\left\{-\frac{a(\theta)}{\varepsilon}\left(1+\frac{\delta}{2}\right)x_j\right\} \quad , \; h_c^2 \le \varepsilon$$

Also

$$\left|g_1^{(3)}(x_j)\right| \le M\left[1+\varepsilon^{-2}\, exp\left\{-\frac{c\,(1+\frac{\delta}{2})}{\varepsilon}x_j\right\}\right] \quad , \; h_c^2 \le \varepsilon$$

and

$$\left|g_2^{(3)}(x_j)\right| \le M\left[1+\varepsilon^{-2}\, exp\left\{-\frac{c\,(1-\frac{\delta}{2})}{\varepsilon}x_j\right\}\right] \quad , \; h_c^2 \le \varepsilon$$

$$\Rightarrow |\tau_j(g_1)| \le M h_c^3\left[1+\frac{1}{\varepsilon^2}\, exp\left\{-\frac{c\,(1+\frac{\delta}{2})}{\varepsilon}x_j\right\}\right] \quad , \; h_c^2 \le \varepsilon$$

and

$$|\tau_j(g_2)| \le M h_c^3\left[1+\frac{1}{\varepsilon^2}\, exp\left\{-\frac{c\,(1-\frac{\delta}{2})}{\varepsilon}x_j\right\}\right] \quad , \; h_c^2 \le \varepsilon$$

Now

$$\tau_j(y) \;=\; \tau_j(g_1)+\tau_j(v)+\tau_j(g_2)$$

$$\Rightarrow \quad |\tau_j(y)| \;\le\; M\frac{h_c^3}{\varepsilon^2}\left[1+exp\left\{-\frac{c\,(1+\frac{\delta}{2})}{\varepsilon}x_j\right\}+exp\left\{-\frac{c\,(1-\frac{\delta}{2})}{\varepsilon}x_j\right\}\right] \quad , \; h_c^2 \le \varepsilon$$

In the opposite case, i.e., when $h_c^2 \ge \varepsilon$, we use the following expression for truncation error :

$$\begin{aligned}\tau_j(y) \;=\; & \left(\frac{h_{j+1}^3}{6}r_j^+ - \frac{h_j^3}{6}r_j^-\right)y'''(\xi_1) + \varepsilon\left(q_j^- h_j - q_j^+ h_{j+1}\right)y'''(\xi_2) \\ & -q_j^-\left\{\frac{h_j^2}{2}a_{j-1}+\frac{h_j^3}{6}b_{j-1}\right\}y'''(\xi_3) - q_j^+\left\{\frac{h_{j+1}^2}{2}a_{j+1}-\frac{h_{j+1}^3}{6}b_{j+1}\right\}y'''(\xi_4)\end{aligned}$$

$$x_{j-1} < \xi_i < x_{j+1}\,,\; i = 1(1)4$$

Now

$$\left|\frac{h_{j+1}^3}{6}r_j^+ - \frac{h_j^3}{6}r_j^-\right| \le M h_c^3 \quad , \quad \left|\varepsilon\left(q_j^- h_j - q_j^+ h_{j+1}\right)\right| \le M h_c^3$$

$$\left|q_j^- \left(\frac{h_j^2}{2}a_{j-1} + \frac{h_j^3}{6}b_{j-1}\right)\right| \le Mh_c^3 \quad , \quad \left|q_j^+ \left(\frac{h_{j+1}^2}{2}a_{j+1} - \frac{h_{j+1}^3}{6}b_{j+1}\right)\right| \le Mh_c^3$$

Using these estimates and the above expression for $\tau_j(y)$, we obtain the same estimates for $\tau_j(g_1)$, $\tau_j(v)$ and $\tau_j(g_2)$ as were in the case of $h_c^2 \le \varepsilon$. Choosing

$$K_1 = h_c^2 exp\left\{-\frac{c\left(1+\frac{\delta}{2}\right)}{\varepsilon}x_j\right\} \quad \text{and} \quad K_2 = h_c^2 exp\left\{-\frac{c\left(1-\frac{\delta}{2}\right)}{\varepsilon}x_j\right\}$$

we see that the Lemma 7.2 is satisfied(in both the cases $h_c^2 \le \varepsilon$ and $h_c^2 \ge \varepsilon$) and thus we have proved the following theorem:

**Theorem 7.2** *Let $\{u_j\}$, $j = 0(1)n$, be a set of values of the approximate solution to $y(x)$ of (7.1), obtained using (7.20) and (7.21). Then there are positive constants $\widetilde{C}_1 \left(= \frac{c(2+\delta)}{2\varepsilon}(1+2\eta C_3)\right)$, $\widetilde{C}_2 \left(= \frac{c(2-\delta)}{2\varepsilon}(1+2\eta C_3)\right)$ and $M$ (independent of $h$ and $\varepsilon$) such that the following estimate holds:*

$$\max_j |y_j - u_j| \le Mh_c^2 \left[exp\left\{-\frac{\widetilde{C}_1 x_j}{\varepsilon}\right\} + exp\left\{-\frac{\widetilde{C}_2 x_j}{\varepsilon}\right\}\right]$$

where

$$h_c = \max_j h_j = \text{a constant.}$$

## 7.6 Test Examples and Numerical Results

In this Section we present some numerical results which illustrate the theory:

**Example 7.1** *[142] : Consider*

$$\varepsilon y'' + 2(1-2x)y' - 4y = 0 \ ; \ y(0) = 1 \ , \ y(1) = 1$$

*whose exact solution is given by*

$$y(x) = exp\left[-\frac{2x(1-x)}{\varepsilon}\right]$$

*Characteristics : The equation has a turning point at $x = 1/2$ and the solution has two boundary layers (one at both ends).*

**Example 7.2** *[142] : Consider*

$$\varepsilon y'' + 2(1-2x)y' - 4y = 4(4x-1) \ ; \ y(0) = 1 \ , \ y(1) = 1$$

*Exact solution is not available.*

*Characteristics : The equation has a turning point at $x = 1/2$ and the solution has two boundary layers (one at both ends).*

**Example 7.3** *[93] : Consider*

$$\varepsilon y'' - xy' = 0 \ ; \ y(-1) = 1 \ , \ y(0.5) = 1.5$$

*Exact solution is not available.*

*Characteristics : The equation has a turning point at $x = 0$ and the solution has a boundary layer of width $O(\varepsilon)$ at the left end.*

**Example 7.4** *[93] : Consider*

$$\varepsilon y'' - xy' = 0 \ ; \ y(-0.5) = 1 \ , \ y(1.5) = 2$$

*Exact solution is not available.*

*Characteristics : The equation has a turning point at $x = 0$ and the solution has a boundary layer of width $O(\varepsilon)$ at the right end.*

**Example 7.5** *[93] : Consider*

$$\varepsilon y'' + xy' - 0.5y = 0 \ ; \ y(-1) = 1 \ , \ y(1) = 2$$

*Exact solution is not available.*

*Characteristics : The equation has a turning point at $x = 0$ and the solution has an interior layer of width $O(\sqrt{\varepsilon})$ in the turning point region.*

**Example 7.6** *[189] : Consider*

$$\varepsilon y'' + xy' = 0 \;\; ; \;\; y(-1) = 0 \;\; , \;\; y(1) = 2$$

*whose exact solution is given by*

$$y(x) = 1 + \frac{erf\left(x/\sqrt{2\varepsilon}\right)}{erf\left(1/\sqrt{2\varepsilon}\right)}$$

*Characteristics : The equation has a turning point at $x = 0$ and the solution has an interior layer of width $O\left(\sqrt{\varepsilon}\right)$ in the turning point region.*

Tables 7.1 to 7.4, 7.19 and 7.20 contain the maximum errors at all the mesh points :

$$\max_j |y_j - u_j|$$

for different $n$ and $\varepsilon$, where $u_j$ is the approximate solution.

Tables 7.5 to 7.12 and 7.16 and 7.17 contain maximum errors based on the double mesh principle (Dollan et al.[66]) (as for Examples 7.2, 7.3, 7.4 and 7.5, the exact solutions are not available) :

$$\max_{0 \le j \le n} \left| u_j^n - u_{2j}^{2n} \right|$$

Table 7.13, 7.14, 7.15, 7.18 and 7.21 contain the numerical rate of uniform convergence which is determined as in [66]:

$$r_{k,\varepsilon} = log_2\left(z_{k,\varepsilon}/z_{k+1,\varepsilon}\right) \;\; , \;\; k = 0, 1, 2, \cdots\cdots\cdots$$

where,

$$z_{k,\varepsilon} = \max_j |u_j^{h_j/2^k} \;\; - \;\; u_{2j}^{h_j/2^{k+1}}| \;\; , \;\; k = 0, 1, 2, \cdots\cdots\cdots$$

and $u_j^{h_j/2^k}$ denotes the value of $u_j$ for the mesh length $h_j/2^k$.

Table 7.1: Max. Errors for Example 7.1
With Uniform Mesh

| $\varepsilon$ | $n = 16$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ |
|---|---|---|---|---|---|---|
| 1/32 | 0.39E+00 | 0.37E-01 | 0.84E-02 | 0.21E-02 | 0.51E-03 | 0.13E-03 |
| 1/64 | 0.67E+00 | 0.14E+00 | 0.36E-01 | 0.81E-02 | 0.20E-02 | 0.49E-03 |
| 1/128 | 0.88E+00 | 0.36E+00 | 0.14E+00 | 0.35E-01 | 0.80E-02 | 0.20E-02 |
| 1/256 | 0.10E+01 | 0.62E+00 | 0.36E+00 | 0.14E+00 | 0.35E-01 | 0.79E-02 |
| 1/512 | 0.84E+01 | 0.80E+00 | 0.61E+00 | 0.35E+00 | 0.14E+00 | 0.35E-01 |
| 1/1024 | 0.33E+02 | 0.91E+00 | 0.79E+00 | 0.60E+00 | 0.35E+00 | 0.14E+00 |
| 1/2048 | 0.92E+02 | 0.97E+00 | 0.90E+00 | 0.78E+00 | 0.60E+00 | 0.35E+00 |
| 1/4096 | 0.22E+03 | 0.10E+01 | 0.95E+00 | 0.89E+00 | 0.78E+00 | 0.60E+00 |

Table 7.2: Max. Errors for Example 7.1
With about 12.5% mesh points in the boundary layer region

| $\varepsilon$ | $n = 16$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ |
|---|---|---|---|---|---|---|
| 1/32 | 0.40E+00 | 0.11E+00 | 0.27E-01 | 0.62E-02 | 0.15E-02 | 0.38E-03 |
| 1/64 | 0.65E+00 | 0.15E+00 | 0.39E-01 | 0.88E-02 | 0.21E-02 | 0.53E-03 |
| 1/128 | 0.16E+01 | 0.19E+00 | 0.53E-01 | 0.12E-01 | 0.29E-02 | 0.72E-03 |
| 1/256 | 0.44E+02 | 0.22E+00 | 0.69E-01 | 0.15E-01 | 0.38E-02 | 0.93E-03 |
| 1/512 | 0.28E+02 | 0.25E+00 | 0.87E-01 | 0.20E-01 | 0.47E-02 | 0.12E-02 |
| 1/1024 | 0.63E+02 | 0.23E+00 | 0.11E+00 | 0.25E-01 | 0.58E-02 | 0.14E-02 |
| 1/2048 | 0.14E+03 | 0.24E+00 | 0.12E+00 | 0.31E-01 | 0.71E-02 | 0.17E-02 |
| 1/4096 | 0.28E+03 | 0.12E+01 | 0.14E+00 | 0.38E-01 | 0.85E-02 | 0.21E-02 |

Table 7.3: Max. Errors for Example 7.1
With about 25% mesh points in the boundary layer region

| $\varepsilon$ | $n = 16$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ |
|---|---|---|---|---|---|---|
| 1/32 | 0.31E+00 | 0.27E-01 | 0.62E-02 | 0.15E-02 | 0.38E-03 | 0.96E-04 |
| 1/64 | 0.25E+00 | 0.39E-01 | 0.88E-02 | 0.22E-02 | 0.54E-03 | 0.13E-03 |
| 1/128 | 0.47E+00 | 0.53E-01 | 0.12E-01 | 0.29E-02 | 0.72E-03 | 0.18E-03 |
| 1/256 | 0.31E+01 | 0.69E-01 | 0.15E-01 | 0.38E-02 | 0.93E-03 | 0.23E-03 |
| 1/512 | 0.17E+02 | 0.87E-01 | 0.20E-01 | 0.47E-02 | 0.12E-02 | 0.29E-03 |
| 1/1024 | 0.54E+02 | 0.11E+00 | 0.25E-01 | 0.58E-02 | 0.14E-02 | 0.36E-03 |
| 1/2048 | 0.14E+03 | 0.12E+00 | 0.31E-01 | 0.71E-02 | 0.17E-02 | 0.43E-03 |
| 1/4096 | 0.31E+03 | 0.14E+00 | 0.38E-01 | 0.85E-02 | 0.21E-02 | 0.52E-03 |

Table 7.4: Max. Errors for Example 7.1
With about 37.5% mesh points in the boundary layer region

| $\varepsilon$ | $n = 16$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ |
|---|---|---|---|---|---|---|
| 1/32 | 0.18E+00 | 0.11E-01 | 0.27E-02 | 0.69E-03 | 0.17E-03 | 0.43E-04 |
| 1/64 | 0.26E+00 | 0.15E-01 | 0.39E-02 | 0.96E-03 | 0.24E-03 | 0.60E-04 |
| 1/128 | 0.45E+00 | 0.22E-01 | 0.51E-02 | 0.13E-02 | 0.32E-03 | 0.80E-04 |
| 1/256 | 0.75E+01 | 0.29E-01 | 0.68E-02 | 0.17E-02 | 0.41E-03 | 0.10E-03 |
| 1/512 | 0.18E+03 | 0.38E-01 | 0.86E-02 | 0.21E-02 | 0.52E-03 | 0.13E-03 |
| 1/1024 | 0.12E+03 | 0.47E-01 | 0.10E-01 | 0.26E-02 | 0.64E-03 | 0.16E-03 |
| 1/2048 | 0.21E+03 | 0.58E-01 | 0.13E-01 | 0.31E-02 | 0.77E-03 | 0.19E-03 |
| 1/4096 | 0.41E+03 | 0.69E-01 | 0.15E-01 | 0.37E-02 | 0.92E-03 | 0.23E-03 |

Table 7.5: Max. Errors for Example 7.2
With Uniform Mesh

| $\varepsilon$ | $n = 32$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ |
|---|---|---|---|---|---|---|
| 1/32 | 0.51E+00 | 0.12E+00 | 0.27E-01 | 0.67E-02 | 0.17E-02 | 0.41E-03 |
| 1/64 | 0.15E+01 | 0.50E+00 | 0.12E+00 | 0.27E-01 | 0.65E-02 | 0.16E-02 |
| 1/128 | 0.32E+01 | 0.15E+01 | 0.49E+00 | 0.12E+00 | 0.26E-01 | 0.64E-02 |
| 1/256 | 0.52E+01 | 0.31E+01 | 0.15E+01 | 0.49E+00 | 0.12E+00 | 0.26E-01 |
| 1/512 | 0.68E+01 | 0.51E+01 | 0.31E+01 | 0.15E+01 | 0.48E+00 | 0.12E+00 |
| 1/1024 | 0.80E+01 | 0.66E+01 | 0.50E+01 | 0.31E+01 | 0.14E+01 | 0.48E+00 |
| 1/2048 | 0.32E+02 | 0.77E+01 | 0.65E+01 | 0.50E+01 | 0.31E+01 | 0.14E+01 |
| 1/4096 | 0.14E+03 | 0.83E+01 | 0.76E+01 | 0.65E+01 | 0.49E+01 | 0.31E+01 |

Table 7.6: Max. Errors for Example 7.2
With about 12.5% mesh points in the boundary layer region

| $\varepsilon$ | $n = 32$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ |
|---|---|---|---|---|---|---|
| 1/32 | 0.14E+01 | 0.38E+00 | 0.85E-01 | 0.19E-01 | 0.42E-02 | 0.85E-03 |
| 1/64 | 0.16E+01 | 0.53E+00 | 0.13E+00 | 0.29E-01 | 0.70E-02 | 0.17E-02 |
| 1/128 | 0.13E+01 | 0.70E+00 | 0.18E+00 | 0.40E-01 | 0.99E-02 | 0.27E-02 |
| 1/256 | 0.19E+01 | 0.86E+00 | 0.24E+00 | 0.51E-01 | 0.13E-01 | 0.36E-02 |
| 1/512 | 0.27E+01 | 0.97E+00 | 0.31E+00 | 0.68E-01 | 0.17E-01 | 0.47E-02 |
| 1/1024 | 0.69E+01 | 0.92E+00 | 0.38E+00 | 0.87E-01 | 0.21E-01 | 0.60E-02 |
| 1/2048 | 0.48E+02 | 0.10E+01 | 0.45E+00 | 0.11E+00 | 0.26E-01 | 0.76E-02 |
| 1/4096 | 0.19E+03 | 0.36E+01 | 0.53E+00 | 0.13E+00 | 0.31E-01 | 0.98E-02 |

Table 7.7: Max. Errors for Example 7.2
With about 25% mesh points in the boundary layer region

| $\varepsilon$ | $n = 32$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ |
|---|---|---|---|---|---|---|
| 1/32 | 0.40E+00 | 0.90E-01 | 0.21E-01 | 0.54E-02 | 0.15E-02 | 0.44E-03 |
| 1/64 | 0.56E+00 | 0.14E+00 | 0.32E-01 | 0.86E-02 | 0.26E-02 | 0.90E-03 |
| 1/128 | 0.73E+00 | 0.19E+00 | 0.44E-01 | 0.12E-01 | 0.37E-02 | 0.13E-02 |
| 1/256 | 0.88E+00 | 0.26E+00 | 0.57E-01 | 0.16E-01 | 0.48E-02 | 0.18E-02 |
| 1/512 | 0.89E+00 | 0.33E+00 | 0.73E-01 | 0.20E-01 | 0.61E-02 | 0.24E-02 |
| 1/1024 | 0.21E+01 | 0.40E+00 | 0.92E-01 | 0.24E-01 | 0.75E-02 | 0.32E-02 |
| 1/2048 | 0.29E+02 | 0.47E+00 | 0.11E+00 | 0.29E-01 | 0.91E-02 | 0.45E-02 |
| 1/4096 | 0.17E+03 | 0.55E+00 | 0.14E+00 | 0.34E-01 | 0.11E-01 | 0.68E-02 |

Table 7.8: Max. Errors for Example 7.2
With about 37.5% mesh points in the boundary layer region

| $\varepsilon$ | $n = 32$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ |
|---|---|---|---|---|---|---|
| 1/32 | 0.19E+00 | 0.44E-01 | 0.12E-01 | 0.39E-02 | 0.15E-02 | 0.70E-03 |
| 1/64 | 0.28E+00 | 0.64E-01 | 0.18E-01 | 0.61E-02 | 0.23E-02 | 0.11E-02 |
| 1/128 | 0.38E+00 | 0.87E-01 | 0.25E-01 | 0.82E-02 | 0.31E-02 | 0.14E-02 |
| 1/256 | 0.49E+00 | 0.12E+00 | 0.32E-01 | 0.10E-01 | 0.39E-02 | 0.18E-02 |
| 1/512 | 0.60E+00 | 0.15E+00 | 0.39E-01 | 0.13E-01 | 0.48E-02 | 0.24E-02 |
| 1/1024 | 0.15E+01 | 0.19E+00 | 0.47E-01 | 0.15E-01 | 0.58E-02 | 0.33E-02 |
| 1/2048 | 0.56E+01 | 0.23E+00 | 0.55E-01 | 0.20E-01 | 0.74E-02 | 0.50E-02 |
| 1/4096 | 0.12E+02 | 0.12E+01 | 0.63E-01 | 0.24E-01 | 0.10E-01 | 0.82E-02 |

Table 7.9: Max. Errors for Example 7.3
With Uniform Mesh

| $\varepsilon$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ | $n = 2048$ |
|---|---|---|---|---|---|---|
| 1/16 | 0.18E-02 | 0.46E-03 | 0.11E-03 | 0.28E-04 | 0.71E-05 | 0.18E-05 |
| 1/32 | 0.70E-02 | 0.17E-02 | 0.42E-03 | 0.10E-03 | 0.26E-04 | 0.65E-05 |
| 1/64 | 0.32E-01 | 0.69E-02 | 0.17E-02 | 0.41E-03 | 0.10E-03 | 0.26E-04 |
| 1/128 | 0.11E+00 | 0.32E-01 | 0.68E-02 | 0.16E-02 | 0.41E-03 | 0.10E-03 |
| 1/512 | 0.17E+01 | 0.55E+00 | 0.22E+00 | 0.64E-01 | 0.14E-01 | 0.33E-02 |
| 1/1024 | 0.17E+01 | 0.98E+00 | 0.54E+00 | 0.22E+00 | 0.64E-01 | 0.14E-01 |
| 1/2048 | 0.15E+01 | 0.10E+01 | 0.97E+00 | 0.54E+00 | 0.22E+00 | 0.64E-01 |
| 1/4096 | 0.14E+01 | 0.17E+01 | 0.15E+01 | 0.97E+00 | 0.54E+00 | 0.22E+00 |

Table 7.10: Max. Errors for Example 7.3
With about 25% mesh points in the boundary layer region

| $\varepsilon$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ | $n = 2048$ |
|---|---|---|---|---|---|---|
| 1/16 | 0.12E-02 | 0.30E-03 | 0.76E-04 | 0.19E-04 | 0.48E-05 | 0.12E-05 |
| 1/32 | 0.34E-02 | 0.95E-03 | 0.25E-03 | 0.65E-04 | 0.16E-04 | 0.41E-05 |
| 1/64 | 0.11E-01 | 0.32E-02 | 0.94E-03 | 0.25E-03 | 0.64E-04 | 0.16E-04 |
| 1/128 | 0.32E-01 | 0.10E-01 | 0.32E-02 | 0.93E-03 | 0.26E-03 | 0.67E-04 |
| 1/512 | 0.22E+01 | 0.51E+00 | 0.17E+00 | 0.40E-01 | 0.90E-02 | 0.22E-02 |
| 1/1024 | 0.13E+01 | 0.12E+01 | 0.51E+00 | 0.17E+00 | 0.40E-01 | 0.89E-02 |
| 1/2048 | 0.19E+01 | 0.39E+01 | 0.11E+01 | 0.50E+00 | 0.17E+00 | 0.40E-01 |
| 1/4096 | 0.67E+00 | 0.41E+01 | 0.58E+01 | 0.11E+01 | 0.50E+00 | 0.17E+00 |

Table 7.11: Max. Errors for Example 7.4
With Uniform Mesh

| $\varepsilon$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ | $n = 2048$ |
|---|---|---|---|---|---|---|
| 1/16 | 0.14E-01 | 0.34E-02 | 0.83E-03 | 0.21E-03 | 0.52E-04 | 0.13E-04 |
| 1/32 | 0.65E-01 | 0.14E-01 | 0.33E-02 | 0.82E-03 | 0.21E-03 | 0.51E-04 |
| 1/64 | 0.22E+00 | 0.64E-01 | 0.14E-01 | 0.33E-02 | 0.82E-03 | 0.20E-03 |
| 1/128 | 0.55E+00 | 0.22E+00 | 0.64E-01 | 0.14E-01 | 0.33E-02 | 0.82E-03 |
| 1/512 | 0.14E+01 | 0.19E+01 | 0.11E+01 | 0.44E+00 | 0.13E+00 | 0.27E-01 |
| 1/1024 | 0.16E+01 | 0.15E+01 | 0.19E+01 | 0.11E+01 | 0.44E+00 | 0.13E+00 |
| 1/2048 | 0.13E+01 | 0.14E+01 | 0.27E+01 | 0.19E+01 | 0.11E+01 | 0.44E+00 |
| 1/4096 | 0.15E+01 | 0.16E+01 | 0.33E+01 | 0.27E+01 | 0.19E+01 | 0.11E+01 |

Table 7.12: Max. Errors for Example 7.4
With about 25% mesh points in the boundary layer region

| $\varepsilon$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ | $n = 2048$ |
|---|---|---|---|---|---|---|
| 1/16 | 0.48E-02 | 0.13E-02 | 0.35E-03 | 0.89E-04 | 0.23E-04 | 0.57E-05 |
| 1/32 | 0.15E-01 | 0.44E-02 | 0.13E-02 | 0.34E-03 | 0.88E-04 | 0.22E-04 |
| 1/64 | 0.43E-01 | 0.14E-01 | 0.43E-02 | 0.13E-02 | 0.34E-03 | 0.88E-04 |
| 1/128 | 0.12E+00 | 0.41E-01 | 0.14E-01 | 0.43E-02 | 0.13E-02 | 0.34E-03 |
| 1/512 | 0.58E+01 | 0.14E+01 | 0.22E+00 | 0.77E-01 | 0.27E-01 | 0.85E-02 |
| 1/1024 | 0.54E+01 | 0.49E+01 | 0.70E+00 | 0.21E+00 | 0.77E-01 | 0.27E-01 |
| 1/2048 | 0.18E+01 | 0.98E+01 | 0.63E+01 | 0.68E+00 | 0.21E+00 | 0.76E-01 |
| 1/4096 | 0.84E+00 | 0.27E+01 | 0.95E+01 | 0.59E+01 | 0.67E+00 | 0.21E+00 |

Table 7.13: Rate of convergence for Example 7.2
With about 25% mesh points in the boundary layer region
$n = 32, 64, 128, 256, 512$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/32 | 0.19E+01 | 0.22E+01 | 0.22E+01 | 0.22E+01 | 0.23E+01 | 0.21E+01 |
| 1/64 | 0.16E+01 | 0.20E+01 | 0.22E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/128 | 0.92E+00 | 0.19E+01 | 0.22E+01 | 0.20E+01 | 0.19E+01 | 0.18E+01 |
| 1/256 | 0.12E+01 | 0.18E+01 | 0.23E+01 | 0.19E+01 | 0.19E+01 | 0.18E+01 |
| 1/512 | 0.15E+01 | 0.17E+01 | 0.22E+01 | 0.20E+01 | 0.18E+01 | 0.18E+01 |
| 1/1024 | 0.29E+01 | 0.13E+01 | 0.21E+01 | 0.21E+01 | 0.18E+01 | 0.20E+01 |
| 1/2048 | 0.56E+01 | 0.12E+01 | 0.21E+01 | 0.21E+01 | 0.18E+01 | 0.25E+01 |
| 1/4096 | 0.57E+01 | 0.28E+01 | 0.20E+01 | 0.21E+01 | 0.17E+01 | 0.29E+01 |

Table 7.14: Rate of convergence for Example 7.3
With about 25% mesh points in the boundary layer region
$n = 32, 64, 128, 256, 512$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/2 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/4 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.19E+01 | 0.20E+01 |
| 1/8 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/16 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/32 | 0.18E+01 | 0.19E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.19E+01 |
| 1/64 | 0.17E+01 | 0.18E+01 | 0.19E+01 | 0.20E+01 | 0.20E+01 | 0.19E+01 |
| 1/128 | 0.17E+01 | 0.17E+01 | 0.18E+01 | 0.18E+01 | 0.20E+01 | 0.18E+01 |
| 1/512 | 0.21E+01 | 0.16E+01 | 0.21E+01 | 0.22E+01 | 0.20E+01 | 0.20E+01 |

Table 7.15: Rate of convergence for Example 7.4
With about 25% mesh points in the boundary layer region
$n = 64, 128, 256, 512, 1024$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/2 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/4 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/8 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/16 | 0.19E+01 | 0.19E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.19E+01 |
| 1/32 | 0.17E+01 | 0.18E+01 | 0.19E+01 | 0.20E+01 | 0.20E+01 | 0.19E+01 |
| 1/64 | 0.16E+01 | 0.17E+01 | 0.18E+01 | 0.19E+01 | 0.19E+01 | 0.18E+01 |
| 1/128 | 0.16E+01 | 0.16E+01 | 0.17E+01 | 0.18E+01 | 0.19E+01 | 0.17E+01 |
| 1/512 | 0.21E+01 | 0.27E+01 | 0.15E+01 | 0.15E+01 | 0.17E+01 | 0.19E+01 |

Table 7.16: Max. Errors for Example 7.5
With Uniform Mesh

| $\varepsilon$ | $n = 32$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ |
|---|---|---|---|---|---|---|
| 1/256 | 0.41E-02 | 0.94E-03 | 0.24E-03 | 0.59E-04 | 0.15E-04 | 0.37E-05 |
| 1/512 | 0.75E-02 | 0.16E-02 | 0.39E-03 | 0.10E-03 | 0.25E-04 | 0.62E-05 |
| 1/1024 | 0.13E-01 | 0.29E-02 | 0.66E-03 | 0.17E-03 | 0.42E-04 | 0.10E-04 |
| 1/2048 | 0.21E-01 | 0.53E-02 | 0.12E-02 | 0.27E-03 | 0.71E-04 | 0.18E-04 |
| 1/4096 | 0.31E-01 | 0.93E-02 | 0.21E-02 | 0.47E-03 | 0.12E-03 | 0.30E-04 |
| 1/8192 | 0.39E-01 | 0.15E-01 | 0.37E-02 | 0.82E-03 | 0.19E-03 | 0.50E-04 |

Table 7.17: Max. Errors for Example 7.5
With about 25% mesh points in the interior layer region

| $\varepsilon$ | $n = 32$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ |
|---|---|---|---|---|---|---|
| 1/256 | 0.12E-02 | 0.29E-03 | 0.76E-04 | 0.19E-04 | 0.47E-05 | 0.12E-05 |
| 1/512 | 0.16E-02 | 0.28E-03 | 0.72E-04 | 0.19E-04 | 0.47E-05 | 0.12E-05 |
| 1/1024 | 0.34E-02 | 0.46E-03 | 0.10E-03 | 0.27E-04 | 0.72E-05 | 0.19E-05 |
| 1/2048 | 0.72E-02 | 0.12E-02 | 0.16E-03 | 0.42E-04 | 0.12E-04 | 0.31E-05 |
| 1/4096 | 0.13E-01 | 0.26E-02 | 0.37E-03 | 0.67E-04 | 0.18E-04 | 0.50E-05 |
| 1/8192 | 0.19E-01 | 0.52E-02 | 0.90E-03 | 0.11E-03 | 0.29E-04 | 0.82E-05 |

Table 7.18: Rate of convergence for Example 7.5
With about 25% mesh points in the interior layer region
$n = 32, 64, 128, 256, 512$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/256 | 0.20E+01 | 0.19E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/512 | 0.25E+01 | 0.20E+01 | 0.19E+01 | 0.20E+01 | 0.20E+01 | 0.21E+01 |
| 1/1024 | 0.29E+01 | 0.22E+01 | 0.19E+01 | 0.19E+01 | 0.19E+01 | 0.22E+01 |
| 1/2048 | 0.26E+01 | 0.29E+01 | 0.19E+01 | 0.19E+01 | 0.19E+01 | 0.22E+01 |
| 1/4096 | 0.23E+01 | 0.29E+01 | 0.24E+01 | 0.19E+01 | 0.19E+01 | 0.23E+01 |
| 1/8192 | 0.19E+01 | 0.25E+01 | 0.30E+01 | 0.19E+01 | 0.18E+01 | 0.22E+01 |

Table 7.19: Max. Errors for Example 7.6
With Uniform Mesh

| $\varepsilon$ | $n = 32$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ |
|---|---|---|---|---|---|---|
| 1/4 | 0.42E-03 | 0.10E-03 | 0.26E-04 | 0.65E-05 | 0.16E-05 | 0.41E-06 |
| 1/8 | 0.79E-03 | 0.20E-03 | 0.50E-04 | 0.12E-04 | 0.31E-05 | 0.78E-06 |
| 1/16 | 0.15E-02 | 0.36E-03 | 0.90E-04 | 0.23E-04 | 0.56E-05 | 0.14E-05 |
| 1/32 | 0.29E-02 | 0.72E-03 | 0.18E-03 | 0.45E-04 | 0.11E-04 | 0.28E-05 |
| 1/64 | 0.52E-02 | 0.14E-02 | 0.36E-03 | 0.90E-04 | 0.22E-04 | 0.56E-05 |
| 1/128 | 0.12E-01 | 0.29E-02 | 0.72E-03 | 0.18E-03 | 0.45E-04 | 0.11E-04 |

Table 7.20: Max. Errors for Example 7.6
With about 25% mesh points in the interior layer region

| | $n = 32$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ | $n = 1024$ |
|---|---|---|---|---|---|---|
| 1/4 | 0.12E-02 | 0.31E-03 | 0.77E-04 | 0.19E-04 | 0.48E-05 | 0.12E-05 |
| 1/8 | 0.13E-02 | 0.34E-03 | 0.84E-04 | 0.21E-04 | 0.52E-05 | 0.13E-05 |
| 1/16 | 0.15E-02 | 0.36E-03 | 0.90E-04 | 0.23E-04 | 0.56E-05 | 0.14E-05 |
| 1/32 | 0.18E-02 | 0.45E-03 | 0.11E-03 | 0.28E-04 | 0.69E-05 | 0.17E-05 |
| 1/64 | 0.30E-02 | 0.67E-03 | 0.16E-03 | 0.41E-04 | 0.10E-04 | 0.25E-05 |
| 1/128 | 0.65E-02 | 0.14E-02 | 0.33E-03 | 0.83E-04 | 0.21E-04 | 0.52E-05 |

Table 7.21: Rate of convergence for Example 7.6
With about 25% mesh points in the interior layer region
$n = 32, 64, 128, 256, 512$

| $\varepsilon$ | r(0) | r(1) | r(2) | r(3) | r(4) | avg |
|---|---|---|---|---|---|---|
| 1/4 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/8 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/16 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/32 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 |
| 1/64 | 0.22E+01 | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.21E+01 |
| 1/128 | 0.23E+01 | 0.21E+01 | 0.20E+01 | 0.20E+01 | 0.20E+01 | 0.21E+01 |

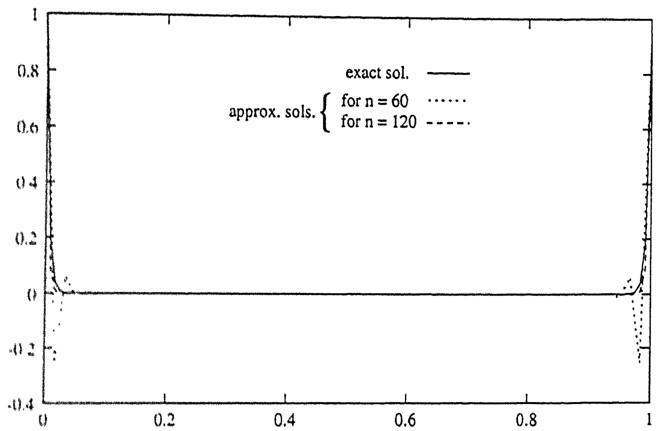


Figure 7.1: Exact and Approx. Solutions of Example 7.1 for $\varepsilon = 0.01$ and $n = 60, 120$ : With Uniform Mesh

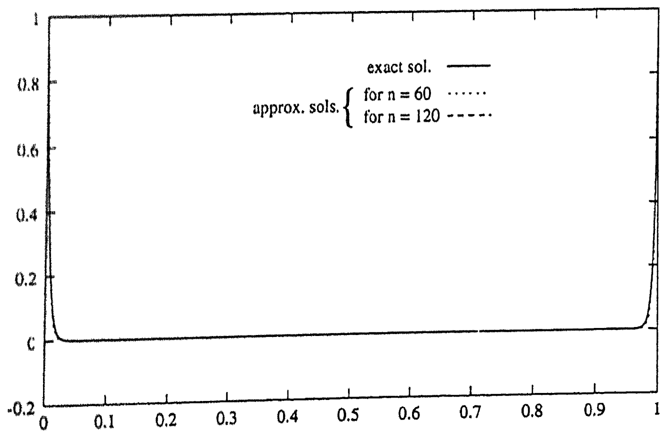


Figure 7.2: Exact and Approx. Solutions of Example 7.1 for $\varepsilon = 0.01$ and $n = 60, 120$ : With about 25 % points in B. Layer

## 7.7 Discussion

We have described a numerical method for solving singularly perturbed turning point problems having boundary/interior layer(s) using cubic spline on non-uniform grid . The method has been analysed for convergence. Several numerical examples have been solved to demonstrate the applicability of the proposed method.

In the given mesh selection strategy, the boundary/interior layer width $\delta$ plays an important role. According to Miller et al. [168], if the solution of the homogeneous singular perturbation problem involves the functions of the type $exp(-x/\varepsilon)$, then $\delta = O(\varepsilon \, ln\,(1/\varepsilon))$ whereas in case the solution involves the functions of the type $exp(-x/\sqrt{\varepsilon}\,)$, then $\delta = O(\sqrt{\varepsilon}\, ln(1/\varepsilon))$. We used this fact while solving Examples 7.1 and 7.2. For Examples 7.1 and 7.2 we took $\delta = O\,(\varepsilon\, ln(1/\varepsilon))$ whereas for Examples 7.3 - 7.4, and for Examples 7.5 - 7.6, we took $\delta = O\,(\varepsilon)$ and $\delta = O\,(\sqrt{\varepsilon})$, respectively, (see [93]).

Our mesh selection procedure needs prior knowledge of $\delta$, $\widetilde{h}_1$ and $K$. We have chosen $\delta$ as above whereas the other two parameters have been taken as $\widetilde{h}_1 = 0.001$ and $K = 0.01$ for Examples 7.1 - 7.2, $\widetilde{h}_1 = 0.00001$ and $K = 1$ for Example 7.3 - 7.6, respectively. However the increase in the value of $K$ will lead to more concentration of points near the boundaries. Moreover, for a fixed $K$, the increase in the value of $\widetilde{h}_1$ leads to the same conclusion.

It can be seen from the Tables 7.13, 7.14 and 7.15 that the rate of convergence for the problems of the type considered in the cases I, II and III is 1.8, 1.9 and 1.7 respectively. Similarly, from the Tables 7.18 and 7.21 it can be observed that the rate of convergence for the Examples 7.5 and 7.6 is 2.

As is seen from the Tables 7.1 to 7.12, 7.16, 7.17, 7.19 and 7.20 the results on non-equidistant grid are better for smaller $\varepsilon$.

To further corroborate the applicability of the proposed method, graphs have been plotted for one example (Example 7.1) for values of $x \in [0,1]$ versus the computed (termed as approximate) solutions obtained at different values of $x$ for a fixed $\varepsilon$. For each plot we took $n = 60$ and 120. Figure 7.1 is the graph, using uniform mesh throughout the region, for Example 7.1 for $\varepsilon = 0.01$, whereas Figure 7.2 is the graph which is plotted using about 25% mesh points in the boundary layer regions (according to the given mesh

selection strategy) for the epsilon value 0.01 and $\delta = O(\varepsilon\, ln(1/\varepsilon))$. It can be seen from Figure 7.1 that the exact and approximate solutions with uniform mesh are identical for most of the range except in the boundary layer regions where these two solutions deviate from each other for smaller $\varepsilon$. To control these fluctuations, we took more mesh points in the boundary layer region and the resulting behaviour can be seen from Figure 7.2 for the corresponding values of $\varepsilon$. The similar observation can be made for other examples also.

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